# Analytical Solutions for the Distribution of Pressure in Permeable Formations with Ellipsoidal Inclusions

**Report RF-97/092**

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| Our reference:<br>**413.11/224042** | Author(s):<br>**Ahmed Sharif** | Version No. / date:<br>**Vers. 1 / 06.06.97** |
| No. of pages:<br>77 | Project Quality Assurance.<br>**Steinar Ekrann** | Distribution restriction:<br>**Open** |
| ISBN:<br>**82-7220-859-8** | Client(s):<br>**NFR-Norwegian Research Council** | |
| Research Program:<br>**RESERVE** | Project title:<br>**Development and Validation of Direct Methods for Two-Phase Upscaling** | |


Scope:

Analytical solutions are developed for the distribution of pressure around a single ellipsoidal inclusion, submerged in a homogeneous matrix of infinite dimensions. The permeability tensors of the inclusion and the surrounding medium are allowed to be arbitrarily oriented relative to the principal axes of the ellipsoid. For the case in which the permeability tensors are aligned with the axes of the ellipsoid, analytical solutions are also developed for composite ellipsoidal inclusions.



Key-words: Pressure Distribution, Ellipsoidal Inclusions, Analytical Solutions, Eccentricity.

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# Summary and Conclusions

Analytical solutions are developed for the distribution of pressure in the vicinity of an ellipsoidal inclusion embedded in an otherwise homogeneous matrix of infinite dimensions. The inclusion and the matrix may both have tensorial permeability. These solutions are needed for the generalization of the self-consistent approximation - an averaging technique recently applied to determine the effective properties (e.g. effective permeability) of heterogeneous media[1, 2].

Two inclusion models were studied. In the first one, the permeability tensors of the ellipsoid and the matrix are both aligned with the principal axes of the ellipsoid. Then the model is generalized (chapter 3) by allowing arbitrary orientation of permeability tensors relative to the principal axes of the ellipsoid and relative to each other. It is found that the pressure fluctuations in the vicinity of the inclusion are described by a set of elliptic integrals. Furthermore, it has been shown that particular analytical solutions reported in the literature are special cases of the generalized solutions presented in this study.

The other model of the study (chapter 4) deals with inclusions made up from a composite ellipsoid, consisting of an interior ellipsoid coated with a confocal ellipsoidal skin of finite thickness. Here we determine the pressure fields associated with an isolated composite inclusion of tensorial permeability, submerged in an embedding matrix of tensorial permeability. In this case, the permeability tensors are aligned with the principal axes of the composite ellipsoids.

The key parameters governing the analytical solutions are: ($i$) permeability tensors of the inclusion and the surrounding medium, ($ii$) eccentricities i.e., the ratios of the shorter semiaxes to the longer semiaxis ($iii$) relative orientation of permeability tensors, ($iv$) volume fraction of the interior ellipsoid (for the composite model only), and ($v$) direction of the uniform field.

Using *Mathematica*, we present in this study contour plots depicting the effects of permeability anisotropy, contrast in permeabilities, geometry of the inclusion, and the relative orientation of permeability tensors, on the distribution of pressure field. For instance, Fig. [A] shows the effect of changing the direction of the inclusion permeability tensor. In this example, the matrix permeability tensor is aligned with the axes of the ellipsoid.

The isobars in Fig. [A] depict the effect of the orientation of permeability tensors on the distribution of potential. In the upper contour plot, both permeability tensors are aligned with the principal axes of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad |x| \le a, |y| \le b, |z| \le c. \tag{A}$$

Hence $\boldsymbol{k}^i = \mathrm{diag}(k^i_x, k^i_y, k^i_z)$ and $\boldsymbol{k}^m = \mathrm{diag}(k^m_x, k^m_y, k^m_z)$. Then we vary the orientation of the permeability tensor of the inclusion by making it diagonal in the coordinate system $(\alpha, \beta, \gamma)$ i.e., $\boldsymbol{k}^i = \mathrm{diag}(k^i_\alpha, k^i_\beta, k^i_\gamma)$, whose orientation relative to the principal axes of the ellipsoid and relative to the permeability tensor $\boldsymbol{k}^m = \mathrm{diag}(k^m_x, k^m_y, k^m_z)$ of the surrounding medium is fixed by the Euler angles $(\pi/2, \pi/4, 3\pi/2)$. The coordinate systems are then related by the following orthogonal transformation

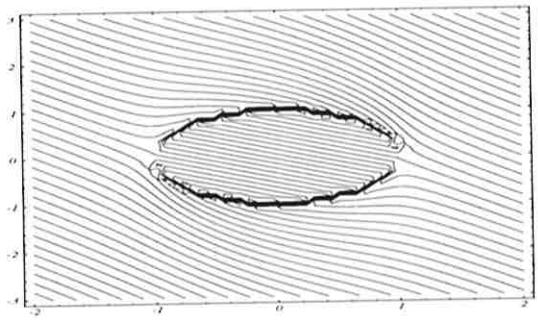
$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \boldsymbol{Q}_i(\pi/2, \pi/4, 3\pi/2) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{B}$$

where

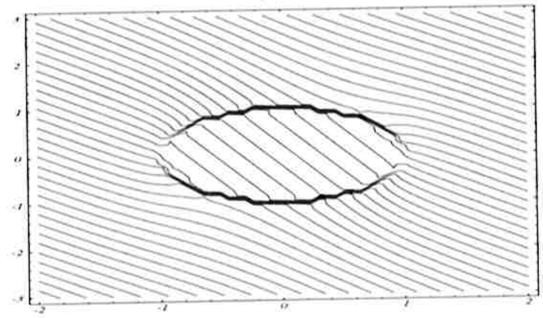
$$\boldsymbol{Q}_i(\pi/2, \pi/4, 3\pi/2) = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}. \tag{C}$$

The distribution of potential in the scaled $(x, z)$-plane (i.e., $x_D = x/a, z_D = z/c$) for both cases are depicted in Fig. [A]. Note that inside the inclusion, the internal potential gradient is constant. In the vicinity of the ellipsoidal surface, the external potential has continuous and twice differentiable gradient. As distances from the inclusion increases, the uniform flow is recovered and the pressure field obeys the prescribed boundary condition.

The main ways in which this study advances on previous work are: first, the permeability tensors of the inclusion and the surrounding matrix are anisotropic, second, we allow arbitrary orientation of permeability tensors relative to the principal axes of the ellipsoid and relative to each other, and finally the utilization of the general ellipsoid provides flexible geometry of the inclusion. Furthermore, the solutions developed here are thought to be very efficient compared to the standard numerical solutions. Therefore, we have strongly generalized the existing inclusion models serving as a basis for the self-consistent method. Hence, a further work that presents itself is to accordingly extend the self-consistent approximation.



(*i*)



(*ii*)

Figure **A**: Isobars in the scaled $(x, z)$-plane: $(i)$ permeability tensors of the inclusion and the surrounding matrix are aligned with the principal axes of the ellipsoid. $(ii)$ permeability tensor of the inclusion is diagonal in the coordinate system whose orientation relative to the principal axes of the ellipsoid is fixed by a 3D counterclockwise rotation of the principal axes of the ellipsoid by the Euler angles $(\pi/2, \pi/4, 3\pi/2)$. Here, the permeability tensor of the matrix is still aligned with the axes of the ellipsoid.

# Chapter 1

# Introduction

The reservoir upscaling problem has been receiving increased attention in recent years. Over the past decade or so, for instance, there has been increasing interest in development of computationally efficient methods to determine effective properties (e.g. effective permeability). Those properties were traditionally computed from detailed numerical simulations of the actual reservoir realization. This is an *indirect* approach, in the terminology of Ekrann *et al.*[2], and it requires substantial computer resources; particularly in 3D problems in which the number of gridblocks often become impractically large. A contrasting strategy is the *direct* approach in which the effective properties are computed directly from the statistical description of the medium - without the aid of an actual reservoir realization. Such methods have the potential to be much less resource intensive than indirect methods, and it will be particularly important for multiphase problems.

Among the direct methods, a particularly promising one which motivated this study is the *self-consistent approximation*. This method which was apparently first deviced by Bruggeman[3] in the context of the determining the electric conductivity of heterogeneous media was later extended to multiphase materials[4, 5]. The method (also termed as the effective medium approximation) has been extended to determine effective hydraulic conductivity of heterogeneous anisotropic formations[1]. In reservoir engineering context, the self-consistent approximation has been recently applied to determine effective permeabilities[2].

The self-consistent approximation needs analytical solutions for the fluctuation of pressure created in an otherwise homogeneous matrix of infinite dimensions by the submersion of inclusions. The existing solutions are based on models which have limitations on the orientation of permeability tensors and perhaps largely in the geometry of the inclusions. Therefore, we develop here analytical solutions for the distribution of pressure by considering the following models

1. **Model 1**: Single ellipsoidal inclusion embedded in an infinite matrix with ($a$) permeability tensors aligned with the principal axes of the ellipsoid, and ($b$) permeability tensors are arbitrarily oriented relative to the principal axes of the ellipsoid.
2. **Model 2**: Orientation of the permeability tensors as in ($1a$), but now the inclusion is made up from a composite ellipsoids in which the ellipsoidal coat between the surface of the interior and of the exterior ellipsoids has a finite thickness.

The above models strongly generalize the existing inclusion models serving as a basis for the self-consistent approximation.

## 1.1 Ellipsoidal Coordinates

The equation

$$S_\alpha : \qquad \frac{x^2}{a^2+\alpha} + \frac{y^2}{b^2+\alpha} + \frac{z^2}{c^2+\alpha} = 1, \tag{1-1}$$

where $a > b > c$ are fixed and $\alpha$ is a parameter, represents a confocal system of surfaces. In particular, if $\alpha = 0$, the equation describes an ellipsoid. For $|\alpha| < \infty$, see Fig. [1-1], the above equation describes

- an ellipsoid if $-c^2 < \alpha < \infty$.
- a hyperboloid of one sheet if $-b^2 < \alpha < -c^2$.
- a hyperboloid of two sheets if $-a^2 < \alpha < -b^2$.
- an imaginary quadric if $\alpha < -a^2$.
- degenerate quadrics if $\alpha = -a^2, -b^2, -c^2$.

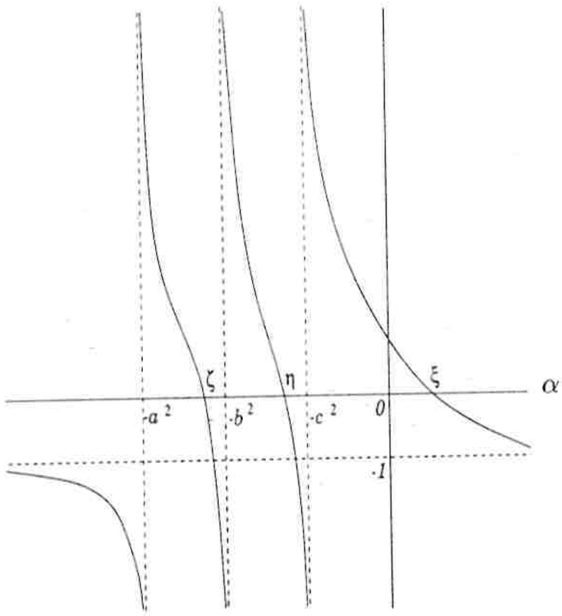



Figure 1-1: *Graph of* $Q(\alpha)q(\alpha) = 0$ *with the* $\xi \geq \eta \geq \zeta$

Assuming the roots are such $\xi \geq \eta \geq \zeta$, the surfaces of the confocal quadrics $\xi =$ const., $\eta =$ const., $\zeta =$ const. are an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets, respectively. These surfaces are described by the following equations

$$S_\xi \; : \qquad \frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1 \qquad -c^2 < \xi < \infty, \tag{1-2}$$

$$S_\eta \; : \qquad \frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} - \frac{z^2}{c^2+\eta} = 1 \qquad -b^2 < \eta < -c^2, \tag{1-3}$$

$$S_\zeta \; : \qquad \frac{x^2}{a^2+\zeta} - \frac{y^2}{b^2+\zeta} - \frac{z^2}{c^2+\zeta} = 1 \qquad -a^2 < \zeta < -b^2. \tag{1-4}$$

Fig. [1-2] depicts surfaces of the ellipsoid $\xi = 0$ and the hyperboloids $\eta = 0$, $\zeta = 0$.

Now, in order to derive the relation between the ellipsoidal and Cartesian coordinates, we may define the functions

$$\begin{aligned} Q(\alpha) &= \frac{x^2}{a^2+\alpha} + \frac{y^2}{b^2+\alpha} + \frac{z^2}{c^2+\alpha} - 1, & (1\text{-}5) \\ q(\alpha) &= (a^2+\alpha)(b^2+\alpha)(c^2+\alpha), & (1\text{-}6) \end{aligned}$$

and observe that the cubic equation

$$\begin{aligned} Q(\alpha)q(\alpha) = x^2(b^2+\alpha)(c^2+\alpha) + y^2(c^2+\alpha)(a^2+\alpha) &+ \\ z^2(a^2+\alpha)(b^2+\alpha) - q(\alpha) &= 0, \end{aligned} \qquad (1\text{-}7)$$

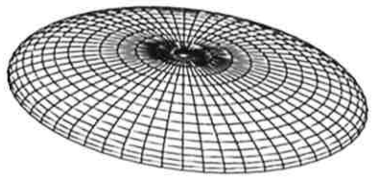
gives the values $\alpha = \xi, \eta, \zeta$ which correspond to the members of the confocal family of surfaces, see Fig. [1-1]. Therefore, since the roots of (1-1) are the zeros of (1-7), we define the identity

$$Q(\alpha)q(\alpha) \equiv (\xi - \alpha)(\eta - \alpha)(\zeta - \alpha), \qquad (1\text{-}8)$$

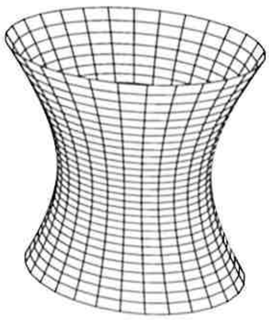
and find that, by setting $\alpha = -a^2$, $-b^2$, $-c^2$ successively, the following formulae provide the required coordinate transformation

$$\begin{aligned} x^2 &= \frac{(a^2+\xi)(a^2+\eta)(a^2+\zeta)}{(a^2-b^2)(a^2-c^2)}, \\ y^2 &= \frac{(b^2+\xi)(b^2+\eta)(b^2+\zeta)}{(b^2-c^2)(b^2-a^2)}, \\ z^2 &= \frac{(c^2+\xi)(c^2+\eta)(c^2+\zeta)}{(a^2-c^2)(b^2-c^2)}. \end{aligned} \qquad (1\text{-}9)$$

It may be observed that the above relations fix the location of, not one, but the eight points $(\pm x, \pm y, \pm z)$ which are symmetrically positioned with respect to the $(x, y, z)$-planes. However, since we will primarily deal with ellipsoidal surfaces only, the problem is symmetric in the $(x, y, z)$-planes. Hence we will not need to distinguish between the symmetric points.



(A)



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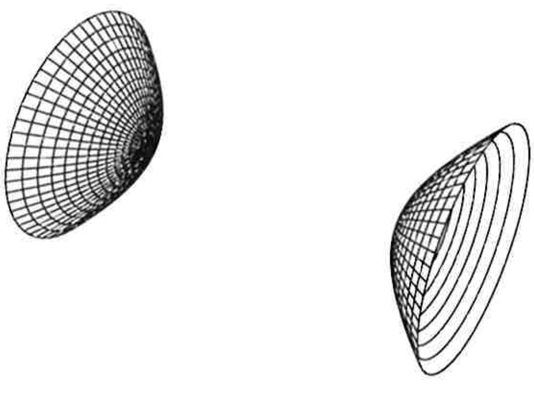


(C)

Figure 1-**2**: *Surfaces of the quadrics: (A) The ellipsoid* $\xi = 0$*, (B) The hyperboloid of one-sheet* $\eta = 0$*, and (C) The hyperboloid of two-sheets* $\zeta = 0$.

### 1.1.1 Description of Inclusion Geometry

Suppose that the axes of an ellipsoidal inclusion are aligned with the principal axes of a coordinate system in which $x$ and $y$ are in the horizontal plane and $z$ is in the vertical plane. Since (1-2) describes the ellipsoidal surfaces, let $\xi = 0$ i.e.,

$$S_0: \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \qquad |x| \le a, |y| \le b, |z| \le c, \tag{1-10}$$

describe the surface of the inclusion. Now, to describe the space interior to $S_0$ in terms of surfaces we may let $\lambda^2 = a^2 + \xi$ for $-c^2 < \xi \le 0$ such that

$$S_\lambda: \qquad \frac{x^2}{\lambda^2 + p^2} + \frac{y^2}{\lambda^2 + q^2} + \frac{z^2}{\lambda^2} = 1; \quad p^2 = a^2 - c^2, \qquad q^2 = b^2 - c^2, \tag{1-11}$$

exclusively describes the ellipsoids $\lambda =$const. inside $S_0$. Note that at the limit $\lambda = 0$, the ellipsoids described by the above equation flatten down to the disc

$$z = 0: \qquad \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \tag{1-12}$$

while for $\lambda^2 = c^2$, the surface equation in (1-10) is recovered. Accordingly, the ellipsoids exterior to $S_0$ will be described by (1-2) for the interval $0 \le \xi < \infty$, and the ellipsoids approximate the spheres $x^2 + y^2 + z^2 \sim \xi$ as $\xi \to \infty$.

Now, let $e_0$ and $E_0$ denote by the eccentricities (i.e., the ratios of the shorter semiaxes to the longer semiaxis) of, in a planetary sense, the *meridian* and *equatorial* ellipses in $S_0$

$$e_0 = \frac{c}{a}, \qquad E_0 = \frac{b}{a} \qquad (a > b > c). \tag{1-13}$$

It is thus obvious that only two parameters are needed to describe the geometry of the ellipsoidal inclusion. For instance, when $e_0 = E_0 \sim 1$, the ellipsoid takes on the form of a slightly perturbed sphere. If, for example, $E_0 = 1$ and $0 < e_0 < 1$ then $a = b > c$ and ( 1-10) describes an *oblate* spheroid; that is the surface of revolution generated by the rotation of an ellipsis about its minor axis. On the other hand, if $E_0 = e_0$, then $b = c < a$ and (1-10) describes a *prolate* spheroid i.e, the surface of revolution generated by the rotation of an ellipsis about its major semiaxis. Furthermore, when either $e_0 \to 0$ or $E_0 \to 0$, then the ellipsoid takes on the form an elliptic cylinder with the $z$- or $y$-axis as the symmetry axis, respectively. Finally, if one of the eccentricities is zero while the other one approaches unity, we have a cylinder.

Consider now an arbitrary ellipsoid exterior to $\xi = 0$. Denoting by $e$ and $E$ the meridian and equatorial eccentricities, we have

$$e = \sqrt{\frac{c^2 + \xi}{a^2 + \xi}}, \qquad E = \sqrt{\frac{b^2 + \xi}{a^2 + \xi}}, \qquad 0 \le \xi < \infty. \tag{1-14}$$

Noting that

$$\frac{a^2 - b^2}{a^2 - c^2} = \frac{(a^2 + \xi) - (b^2 + \xi)}{(a^2 + \xi) - (c^2 + \xi)} \qquad \text{for} \quad \xi \ge 0, \tag{1-15}$$

define

$$\frac{a^2 - b^2}{a^2 - c^2} = \kappa^2. \tag{1-16}$$

Using the definition of eccentricities, we obtain

$$\frac{1-E_0^2}{1-e_0^2} = \frac{1-E^2}{1-e^2} = \kappa^2. \qquad (1\text{-}17)$$

Hence, by the relation

$$E^2 + \kappa^2(1-e^2) = 1 \qquad \text{(for fixed } \xi) \qquad (1\text{-}18)$$

we may operate with either one of the eccentricities. Fig. [1-3] shows the relation between the eccentricities of a general ellipsoid

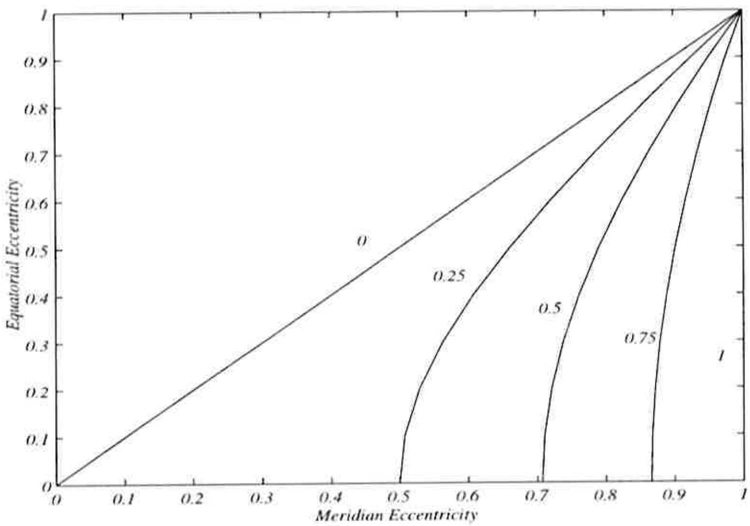



Figure 1-3: *Relatioship between the meridian & equatorial eccentricities of general ellipsoids at various values of* $\kappa^2$.

## 1.2 Brief Theory of Ellipsoidal Harmonics

Consider the function $\Phi(x, y, z)$ satisfied by the Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \qquad (1\text{-}19)$$

and assume that $\Phi(x, y, z)$ is distributed inside and outside the surface of the ellipsoid $\xi$ =const., in the presence of a certain boundary condition. This type of problem is mainly treated in the classical works on applied mathematics[8, 9], potential theory[11] and hydrodynamics[12, 13].

A straightforward procedure of solving (1-19) may read as follows: let $\Phi_1$ and $\Phi_2$ be the basis of a solution of ( 1-19), where $\Phi_1$ is obtained by guessing (or by some other method). Provided the boundary condition satisfied by $\Phi$, derive a second linearly independent basis of the form

$$\Phi(x, y, z) = \Gamma(x, y, z)\Phi_1(x, y, z), \qquad (1\text{-}20)$$

by determining the function $\Gamma(x, y, z)$ such that a possible solution of (1-19) is constructed by superposition

$$\Phi(x,y,z) = c_1\Phi_1(x,y,z) + c_2\Phi_2(x,y,z) \qquad c_1, c_2 \text{ arbitrary constants.} \tag{1-21}$$

Since the position of a point $P(x,y,z)$ can be fixed by a set a values for the ellipsoidal coordinates $(\xi,\eta,\zeta)$, it may henceforth be convenient to consider solving the Laplace equation in ellipsoidal coordinates (see Appendix A)

$$(\eta-\zeta)D(\xi)\frac{\partial}{\partial\xi}\left[D(\xi)\frac{\partial\Phi}{\partial\xi}\right] + (\zeta-\xi)D(\eta)\frac{\partial}{\partial\eta}\left[D(\eta)\frac{\partial\Phi}{\partial\eta}\right] + (\xi-\eta)D(\zeta)\frac{\partial}{\partial\zeta}\left[D(\zeta)\frac{\partial\Phi}{\partial\zeta}\right] = 0 \tag{1-22}$$

where

$$D(\alpha) = \sqrt{(a^2+\alpha)(b^2+\alpha)(c^2+\alpha)}\,, \qquad \alpha = \xi,\eta,\zeta. \tag{1-23}$$

Solutions of (1-22) are known as the *ellipsoidal harmonics.*

### 1.2.1 Internal Ellipsoidal Harmonics

By certain choice of the constants required for the separation of variables in (1-22), of the form

$$\Phi(\xi,\eta,\zeta) = L(\xi)M(\eta)N(\zeta), \tag{1-24}$$

we obtain a set of normal equations known as Lamé's differential equations. Solutions of such equations which are known as Lamé's functions may form the basis for the ellipsoidal harmonics. Now, an extensive analysis of the possible solutions for Lamé's equation is beyond the scope of this study (see [8, 9, 10] for detailed discussion) and we do not pursue it here. There are, however, three points of particular relevance for the construction of ellipsoidal harmonics:

1. Lamé's functions of degree can be represented by a terminating series of Legendre polynomial. Hence the internal ellipsoidal harmonics of degree $n$ can be represented by a sum of $2n+1$ spherical harmonics.
2. The choice of $L(\xi)$ depends on the type of ellipsoidal harmonics under consideration[10]. For an *internal* ellipsoidal harmonics (Lamé's first solution), the product in $LMN$ is required to be regular inside the space bounded by the ellipsoid $\xi = $ const., while for the *external* harmonics (Lamé's second solution), an additional requirement is solutions which vanish at infinity i.e, as $\xi \to \infty$. For the case of an ellipsoidal harmonic regular between two ellipsoids of a confocal family, we may take a linear combination of the internal and external ellipsoidal harmoninics.
3. To every internal ellipsoidal harmonic there corresponds an external ellipsoidal harmonic which satisfy the infinity condition mentioned in (2).

Switching to the Cartesian coordinates, it follows from (1) that 1, $x$, $xy$, $xyz$ and in fact any other spherical harmonic may form the basis for a possible internal ellipsoidal harmonic. Here 1 is an *ellipsoidal harmonic of the first species*, $x, y, z$ give an *ellipsoidal harmonic of the second species*, $xy, xz, yz$ give an *ellipsoidal harmonic of the third species* and $xyz$ is an *ellipsoidal harmonic of the fourth species.* Thus, by choosing any particular internal ellipsoidal harmonic, we may accordingly derive the basis for the corresponding external harmonic by employing the conventional methods of determining a second basis when the first is 'known'. This is the strategy we will utilize.

### 1.2.2 External Ellipsoidal Harmonics

For the distribution of $\Phi$ in the space exterior to surface of a certain ellipsoid, each member of the family of the confocal ellipsoids $\xi$ =const., would be equipotential[11]. Therefore, we make $\Gamma$ dependent on the ellipsoidal surface parameter only, such that, by the trial basis in (1-20)

$$\Phi(\xi,\eta,\zeta)=\Gamma(\xi)\Phi_1(\xi,\eta,\zeta). \tag{1-25}$$

Differentiation of the above gives

$$\begin{aligned}
\frac{\partial}{\partial\xi}\left[D(\xi)\frac{\partial}{\partial\xi}(\Gamma\Phi_1)\right] &= \Gamma\frac{\partial}{\partial\xi}[D(\xi)\frac{\partial\Phi_1}{\partial\xi}]+D(\xi)\frac{\partial\Phi_1}{\partial\xi}\frac{\partial\Gamma}{\partial\xi}+\frac{\partial}{\partial\xi}[\Phi_1 D(\xi)\frac{\partial\Gamma}{\partial\xi}],\\
\frac{\partial}{\partial\eta}\left[D(\eta)\frac{\partial}{\partial\eta}(\Gamma\Phi_1)\right] &= \Gamma\frac{\partial}{\partial\eta}[D(\eta)\frac{\partial\Phi_1}{\partial\eta}],\\
\frac{\partial}{\partial\zeta}\left[D(\zeta)\frac{\partial}{\partial\zeta}(\Gamma\Phi_1)\right] &= \Gamma\frac{\partial}{\partial\zeta}[D(\zeta)\frac{\partial\Phi_1}{\partial\zeta}].
\end{aligned} \tag{1-26}$$

Substitution into (1-22) leads to

$$\Gamma\nabla^2\Phi_1+(\eta-\zeta)D(\xi)\left\{\frac{\partial}{\partial\xi}\left[\Phi_1 D(\xi)\frac{\partial\Gamma}{\partial\xi}\right]+D(\xi)\frac{\partial\Phi_1}{\partial\xi}\frac{\partial\Gamma}{\partial\xi}\right\}=0. \tag{1-27}$$

Since $\Phi_1$ is a solution of (1-22), the first term in the above equation vanishes and we obtain

$$\frac{\partial}{\partial\xi}\ln\left[D(\xi)\frac{\partial\Gamma}{\partial\xi}\right]=-\frac{2}{\Phi_1}\frac{\partial\Phi_1}{\partial\xi}. \tag{1-28}$$

Furthermore, since the left side of this equation is a function of $\xi$ only, $\Phi_1$ must be separable in the form

$$\Phi_1(\xi,\eta,\zeta)=\chi(\xi)\psi(\eta,\zeta). \tag{1-29}$$

Hence (1-28) is reduced to the ordinary differential equation

$$\frac{d}{d\xi}\ln\left[D(\xi)\frac{d\Gamma}{d\xi}\right]=\frac{d}{d\xi}\ln\frac{1}{\chi^2}, \tag{1-30}$$

which, on integration, gives

$$\Gamma(\xi|\chi)=c_0\int_\xi^\infty\frac{du}{\chi^2(u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \tag{1-31}$$

where $c_0$ is an integration constant and the additive term is chosen so as to make $\Gamma$ vanish at infinity. We call the above integral the *associated exterior harmonics*, and

$$\Phi_2=\Phi_1\int_\xi^\infty\frac{du}{\chi^2(u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \tag{1-32}$$

the basis for an *external harmonic*. Accordingly, a possible solution of (1-19) is given by the ellipsoidal harmonic

$$\Phi=c_1\Phi_1+c_2\Phi_1\int_\xi^\infty\frac{du}{\chi^2(u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \tag{1-33}$$

Obviously, the behaviour (e.g. singularity, convergence, analyticity) of the second basis in the above solution is too much reliant upon the choice of $\Phi_1$; largely by means of $\chi$. The following table shows $\chi$ for the spherical harmonics[1].

| $\Phi_1$ | $\chi^2(\xi)$ |
|---|---|
| 1 | 1 |
| $x$ | $(a^2+\xi)$ |
| $y$ | $(b^2+\xi)$ |
| $z$ | $(c^2+\xi)$ |
| $xy$ | $(a^2+\xi)(b^2+\xi)$ |
| $xz$ | $(a^2+\xi)(c^2+\xi)$ |
| $yz$ | $(b^2+\xi)(c^2+\xi)$ |

Table 1.1: $\chi^2$ *for the ellipsoidal harmonics of first, second and third species*

Fig.[1-4] depicts plots of associated exterior harmonics which are derived from the ellipsoidal harmonics of 2nd and 3rd species.

Finally, we consider the computational aspects of the integral in (1-33). Obviously, we will need change of variables to transform the integration interval in (1-31) into a finite one. Therefore, we use the transformation

$$\cos^2\vartheta = \frac{c^2+\xi}{a^2+\xi}, \tag{1-34}$$

such that the integral in (1-31) becomes

$$\Gamma(\varphi|\kappa) = \frac{2c_0}{\sqrt{a^2-c^2}}\int_0^{\varphi}\frac{d\phi}{\chi^2(\phi)\sqrt{1-\kappa^2\sin^2\phi}}, \tag{1-35}$$

where $\kappa$ is defined[2] in (1-16). For the ellipsoidal harmonics of first and second species, $\chi^2$ has the forms given by the tabel

| $\Phi_1$ | $\chi^2(\varphi)$ |
|---|---|
| 1 | 1 |
| $x$ | $(a^2-c^2)(1-\kappa^2)\csc^2\varphi$ |
| $y$ | $(a^2-c^2)[1-\kappa^2\sin^2\varphi]\csc^2\varphi$ |
| $z$ | $(a^2-c^2)(1-\sin^2\varphi)\csc^2\varphi$ |

Table 1.2: $\chi^2$ *for ellipsoidal harmonics of first and second species*

It may be observed that if $\Phi_1$ is chosen such that $\chi = 1$, then

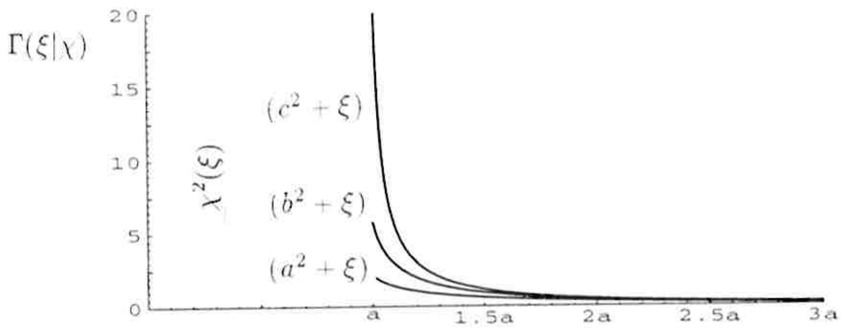
$$\Gamma(\varphi|\kappa) = \frac{2c_0}{\sqrt{a^2-c^2}}F(\varphi|\kappa); \qquad F(\varphi|\kappa) \equiv \int_0^{\varphi}\frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}, \tag{1-36}$$

[1] For convenience, we put $c_4 = c_5 = 0$ in (??)

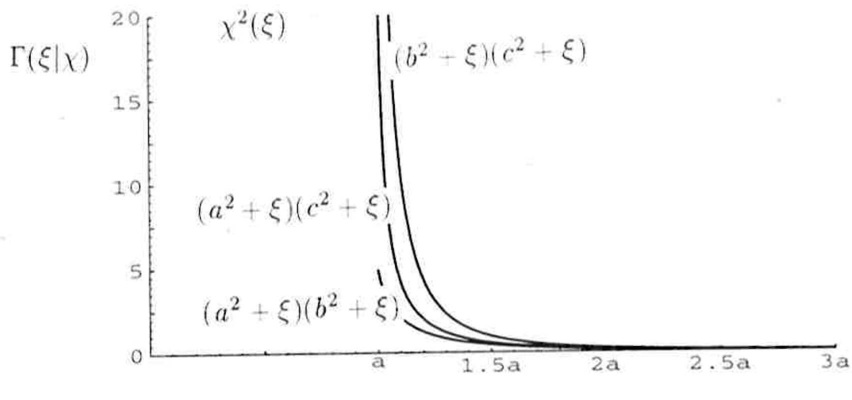
[2] In the terminology of pendulum mechanics, $\vartheta$ denotes by the amplitude, $\kappa$ the modulus.

where $F$ denotes the Jacobian form of the *incomplete* elliptic integral of the first kind[16]. Since any elliptic integral can be expressed in terms of the three Legendre-Jacobi elliptic integrals[16, 17], we will show in Chapter 2 that the associated exterior harmonics can be made dependent upon the first and second kinds of those standard elliptic integrals, and does not involve those of the third kind. The standard elliptic integral of the second is defined

$$E(\varphi|\kappa) \equiv \int_0^{\varphi} \sqrt{1 - \kappa^2 \sin^2 \phi}\, d\phi. \tag{1-37}$$




(*A*)




(*B*)

Figure 1-4: *Plots of Associated External Harmonics as a function of ellipsoidal surface parameter.* (***A***) $\Phi_1$ *is constructed from internal harmonics of second species, (i.e.,* $x, y, z$*).* (***B***) $\Phi_1$ *is constructed from internal ellipsoidal harmonics of third species (i.e.,* $xy, xz, yz$*)*

# Chapter 2

# Single Ellipsoidal Inclusions

In this chapter, we consider single-phase incompressible steady flow in a heterogeneous medium made up from the submersion of an isolated single[1] ellipsoidal inclusion in a homogeneous matrix of infinite dimensions. The permeability tensors of the inclusion and the surrounding matrix are aligned with the principal axes of the ellipsoid, see Fig. [2-1].

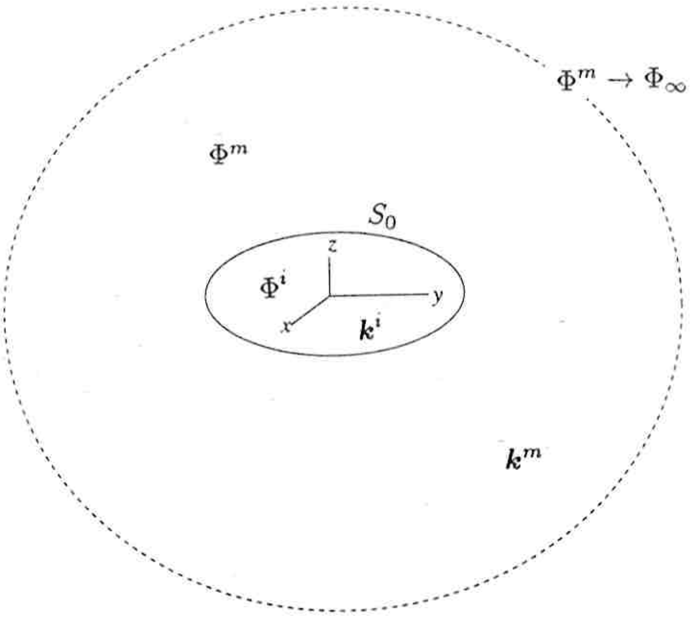



Figure 2-1: *Ellipsoidal inclusion of permeability* $\boldsymbol{k}^i$ *submerged in a uniform matrix of permeability* $\boldsymbol{k}^m$. *Permeability tensors are aligned with the ellipsoid.*

## 2.1 Formulation of the Problem

Let $\Omega \subset R^3$ be a flow domain of infinite dimensions. Denoting by $S_0$ the surface of the ellipsoidal inclusion, let $\Omega_0$ and $\Omega_1$ be a partition of $\Omega$ that excludes the surface of the ellipsoid $S_0$, of equation

[1] We shall use the terms *single* and *composite* ellipsoidal inclusion to geometrically distinguish the two models studied here.

$$S_0: \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1; \qquad \boldsymbol{D} = \mathrm{diag}(1/a^2, 1/b^2, 1/c^2), \tag{2-1}$$

where $a > b > c$ are the semi-axes of the ellipsoid.

### Go erning Equations

The fundamental equation governing the flow of fluid in each region in $\Omega$ may be derived from the equation of continuity and Darcy's law

$$\nabla \cdot \boldsymbol{u}^\sigma = 0, \qquad \mu \boldsymbol{u}^\sigma = -\boldsymbol{k}^\sigma \nabla \Phi^\sigma \qquad (\sigma\text{: inclusion, matrix}). \tag{2-2}$$

Here $\mu$ is the fluid viscosity, $\boldsymbol{u}$ is the fluid velocity and $\Phi$ is the potential. By the above equations, we obtain the following second-order elliptic partial differential equation

$$\nabla \cdot (\boldsymbol{k}^\sigma \nabla \Phi^\sigma) = 0 \qquad (\sigma\text{: inclusion, matrix}). \tag{2-3}$$

Assuming local homogeneity of the inclusion and the surrounding medium, the field potentials satisfy the equations

$$\boldsymbol{k}^i \nabla^2 \Phi^i = 0 \qquad \text{(inside } S_0\text{)}, \tag{2-4}$$

$$\boldsymbol{k}^m \nabla^2 \Phi^m = 0 \qquad \text{(outside } S_0\text{)}. \tag{2-5}$$

Furthermore, here we consider a case in which the permeability tensors are aligned with the principal axes of the ellipsoid i.e,

$$\boldsymbol{k}^i = \mathrm{diag}(k_x^i, k_y^i, k_z^i); \qquad \boldsymbol{k}^m = \mathrm{diag}(k_x^m, k_y^m, k_z^m). \tag{2-6}$$

Therefore, signifying by $\sigma = i, m$ the inclusion and the matrix parameters, the internal and external potentials satisfy read

$$k_x^i \frac{\partial^2 \Phi^i}{\partial x^2} + k_y^i \frac{\partial^2 \Phi^i}{\partial y^2} + k_z^i \frac{\partial^2 \Phi^i}{\partial z^2} = 0 \qquad \text{in } \Omega_0, \tag{2-7}$$

$$k_x^m \frac{\partial^2 \Phi^m}{\partial x^2} + k_y^m \frac{\partial^2 \Phi^m}{\partial y^2} + k_z^m \frac{\partial^2 \Phi^m}{\partial z^2} = 0 \qquad \text{in } \Omega_1. \tag{2-8}$$

### Surface Conditions

Assuming $S_0$ is sharp, two flow conditions have to be obeyed at every surface point. First, the continuity of potential requires

$$\Phi_P^i = \Phi_P^m, \tag{2-9}$$

where $P \in S_0$ is an arbitrary surface point. Similarly, the normal component of the flux is required to be continuous across $S_0$

$$\left(\boldsymbol{k}^i \frac{\partial \Phi^i}{\partial n}\right)_P = \left(\boldsymbol{k}^m \frac{\partial \Phi^m}{\partial n}\right)_P \tag{2-10}$$

across the surface $S_0$. Here $n$ denotes the outward normal to $P \in S_0$.

### Boundary Condition

Finally, we conclude the formulation of the problem by assuming uniform flow at infinity. Thus, denoting by $\Phi_\infty$ the far-field potential, condition

$$\Phi^m(x,y,z) \rightarrow \Phi_\infty(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{as} \quad \|(x,y,z)\| \rightarrow \infty, \tag{2-11}$$

where

$$\boldsymbol{J}^T = (J_x, J_y, J_z) \tag{2-12}$$

## 2.2 Solution of the Potential Problems

### 2.2.1 Fundamental Assumptions

Utilizing appropriate coordinate transformations, we shall solve the isotropic equivalents of the potential problems in (2-7) and (2-8) by applying the solution methodology developed in Section 1.2, by making the following fundamental assumptions:

- *Ellipsoidal harmonics of second species constitute the basis for a possible solution.*
- *Distribution of potential is linear in* $\boldsymbol{J}$.

The above assumptions imply that ($i$) the distribution of potential inside the inclusion is linear, and ($ii$) possible solution of the problems can be determined by superposition of separate solutions in $\boldsymbol{J}$.

### 2.2.2 Solution of the Internal Problem

Introducing the following coordinate transformation,

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \boldsymbol{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad \boldsymbol{V} = \text{diag}\left(\sqrt{\frac{k^i}{k^i_x}}, \sqrt{\frac{k^i}{k^i_y}}, \sqrt{\frac{k^i}{k^i_z}}\right), \tag{2-13}$$

where $k^i$ may be chosen arbitrarily[2], the isotropic equivalent of (2-7) reads

$$\frac{\partial^2 \Phi^i}{\partial \tilde{x}^2} + \frac{\partial^2 \Phi^i}{\partial \tilde{y}^2} + \frac{\partial^2 \Phi^i}{\partial \tilde{z}^2} = 0 \qquad \text{in } \tilde{\Omega}_0 , \tag{2-14}$$

where $\tilde{\Omega}_0$ is the transform of $\Omega_0$ described here by the surface parameter $\tilde{\lambda}$ (see equation (1-11)) for $0 < \tilde{\lambda}^2 \leq \tilde{c}^2$, such that for every internal point

$$\tilde{S}_0 : \qquad \frac{\tilde{x}^2}{\tilde{\lambda}^2 + \tilde{p}} + \frac{\tilde{y}^2}{\tilde{\lambda}^2 + \tilde{q}} + \frac{\tilde{z}^2}{\tilde{\lambda}^2} = 1; \qquad \tilde{p} = \tilde{a}^2 - \tilde{c}^2; \qquad \tilde{q} = \tilde{b}^2 - \tilde{c}^2, \tag{2-15}$$

[2] By requiring that the volume of the ellipsoid is preserved in transformations, we put $k^i = (k^i_x k^i_y k^i_z)^{\frac{1}{3}}$ and $k^m = (k^m_x k^m_y k^m_z)^{\frac{1}{3}}$.

where

$$\begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{bmatrix} = \boldsymbol{V} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \text{such that} \quad \tilde{\boldsymbol{D}} = \boldsymbol{V}^{-2}\boldsymbol{D}. \tag{2-16}$$

We now refer to our discussion on the internal ellipsoidal harmonics (see §1.2). Since we assume that the ellipsoidal harmonics of the second species constitute the basis for a possible solution to the problem in (2-14), a possible solution of the distribution of pressure inside the inclusion is

$$\Phi^i(\tilde{x}, \tilde{y}, \tilde{z}) = \boldsymbol{v}^T \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \qquad (\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{\Omega}_0 \tag{2-17}$$

where $\boldsymbol{v}$ is a constant vector to be determined by the continuity conditions in (2-9) and (2-10). Switching to the $(x, y, z)$ coordinate system, the distribution of pressure in the space bounded by the surface $S_0$ is

$$\Phi^i(x, y, z) = \boldsymbol{v}^T \boldsymbol{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in \Omega_0 \tag{2-18}$$

where $\boldsymbol{V}$ is the diagonal matrix defined in (2-13).

### 2.2.3 Solution of the External Problem

In a similar procedure, we make use of the coordinate transformation

$$\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = \boldsymbol{W} \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad \boldsymbol{W} = \text{diag}\left(\sqrt{\frac{k^m}{k^m_x}}, \sqrt{\frac{k^m}{k^m_y}}, \sqrt{\frac{k^m}{k^m_z}}\right), \tag{2-19}$$

such that the isotropic equivalents of (2-8) and (2-11) read

$$\frac{\partial^2 \Phi^m}{\partial \acute{x}^2} + \frac{\partial^2 \Phi^m}{\partial \acute{y}^2} + \frac{\partial^2 \Phi^m}{\partial \acute{z}^2} = 0 \qquad \text{in } \acute{\Omega}_1 \ , \tag{2-20}$$

$$\Phi^m(\acute{x}, \acute{y}, \acute{z}) \to \Phi_\infty(\acute{x}, \acute{y}, \acute{z}) = -\acute{\boldsymbol{J}}^T \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} \qquad \text{as} \quad \|(\acute{x}, \acute{y}, \acute{z})\| \to \infty. \tag{2-21}$$

where $\acute{\Omega}_1$ is the transform of the exterior flow domain, and

$$\acute{\boldsymbol{J}} = \boldsymbol{W}^{-1}\boldsymbol{J}. \tag{2-22}$$

Accordingly, denoting by $\acute{S}_0$ the transform of $S_0$, the equation of the ellipsoid in the new coordinates is

$$\acute{S}_0 : \qquad \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix}^T \acute{\boldsymbol{D}} \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = 1; \qquad \acute{\boldsymbol{D}} = \text{diag}(1/\acute{a}^2, 1/\acute{b}^2, 1/\acute{c}^2), \tag{2-23}$$

where

$$\begin{bmatrix} \acute{a} \\ \acute{b} \\ \acute{c} \end{bmatrix} = \boldsymbol{W} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \acute{\boldsymbol{D}} = \boldsymbol{W}^{-2}\boldsymbol{D}. \tag{2-24}$$

Based on the preceding assumptions on the linearity of potential and the possible basis for the distribution of pressure, it is immediately seen that the exterior problem in (2-20) admits the following formulation: assuming that the ellipsoidal harmonics of second species (see 1.2.2) constitute the basis of a possible solution to (2-20), let

$$\Phi_1^m(\acute{x}, \acute{y}, \acute{z}) = \boldsymbol{c}^T \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} \qquad \boldsymbol{c} \text{ arbitrary constant vector} \qquad (\text{inside } \acute{S}_0), \tag{2-25}$$

determine the function $\Gamma(\acute{x}, \acute{y}, \acute{z})$ such that

$$\Phi^m(\acute{x}, \acute{y}, \acute{z}) = c_1\Phi_1^m(\acute{x}, \acute{y}, \acute{z}) + c_2\Gamma(\acute{x}, \acute{y}, \acute{z})\Phi_1^m(\acute{x}, \acute{y}, \acute{z}) \qquad (c_1, c_2 \text{ arbitrary constants}), \tag{2-26}$$

which obeys the far-field condition in (2-21), is a possible solution to (2-20).

To solve the above problem by straightforward reference to the solution methodology developed in §1.2.2, we proceed as follows. First, in virtue of the transformation in (2-19), let $\acute{\xi}, \acute{\eta}, \acute{\zeta}$ be the transforms of $\xi, \eta, \zeta$ which are accordingly related to their Cartesian counterparts, $\acute{x}, \acute{y}, \acute{z}$ through the transformations (see Section 1.1)

$$\begin{aligned} \acute{x}^2 &= \frac{(\acute{a}^2+\acute{\xi})(\acute{a}^2+\acute{\eta})(\acute{a}^2+\acute{\zeta})}{(\acute{a}^2-\acute{b}^2)(\acute{a}^2-\acute{c}^2)}, \\ \acute{y}^2 &= \frac{(\acute{b}^2+\acute{\xi})(\acute{b}^2+\acute{\eta})(\acute{b}^2+\acute{\zeta})}{(\acute{b}^2-\acute{c}^2)(\acute{b}^2-\acute{a}^2)}, \\ \acute{z}^2 &= \frac{(\acute{c}^2+\acute{\xi})(\acute{c}^2+\acute{\eta})(\acute{c}^2+\acute{\zeta})}{(\acute{a}^2-\acute{c}^2)(\acute{b}^2-\acute{c}^2)}. \end{aligned} \tag{2-27}$$

Next, by utilizing the above coordinate transformation, the problem in (2-20) in terms of ellipsoidal coordinates reads (see Appendix A)

$$(\acute{\eta}-\acute{\zeta})\acute{D}(\acute{\xi})\frac{\partial}{\partial\acute{\xi}}\left[\acute{D}(\acute{\xi})\frac{\partial\Phi^m}{\partial\acute{\xi}}\right] + (\acute{\zeta}-\acute{\xi})\acute{D}(\acute{\eta})\frac{\partial}{\partial\acute{\eta}}\left[\acute{D}(\acute{\eta})\frac{\partial\Phi^m}{\partial\acute{\eta}}\right] + (\acute{\xi}-\acute{\eta})\acute{D}(\acute{\zeta})\frac{\partial}{\partial\acute{\zeta}}\left[\acute{D}(\acute{\zeta})\frac{\partial\Phi^m}{\partial\acute{\zeta}}\right] = 0. \tag{2-28}$$

Here

$$\acute{D}(\acute{\alpha}) = \sqrt{(\acute{a}^2+\acute{\alpha})(\acute{b}^2+\acute{\alpha})(\acute{c}^2+\acute{\alpha})}\,, \qquad \acute{\alpha} = \acute{\xi}, \acute{\eta}, \acute{\zeta}. \tag{2-29}$$

Furthermore, by the expressions in (2-27), we have

$$\acute{x} = f(\acute{\eta}, \acute{\zeta})\sqrt{\acute{a}^2+\acute{\xi}}, \qquad \acute{y} = g(\acute{\eta}, \acute{\zeta})\sqrt{\acute{b}^2+\acute{\xi}}, \qquad \acute{z} = h(\acute{\eta}, \acute{\zeta})\sqrt{\acute{c}^2+\acute{\xi}}. \tag{2-30}$$

Hence $\Phi_1^m$ satisfies the separation condition (see equation (1-29))

$$\Phi_1^m(\acute{\xi}, \acute{\eta}, \acute{\zeta}) = \acute{\chi}(\acute{\xi})\acute{\psi}(\acute{\eta}, \acute{\zeta}), \tag{2-31}$$

Finally, since each family of the confocal ellipsoids $\acute{\xi}$ = const. would be equipotential[11], a second basis of the form

$$\Phi^m(\acute{\xi}, \acute{\eta}, \acute{\zeta}) = \Gamma(\acute{\xi})\Phi_1^m(\acute{\xi}, \acute{\eta}, \acute{\zeta}). \tag{2-32}$$

leads to the following elliptic integral, (refer to §1.2.2)

$$\Gamma(\acute{\xi}|\acute{\chi}) = c_0 \int_{\acute{\xi}}^{\infty} \frac{d\acute{u}}{\acute{\chi}^2(\acute{u})\sqrt{(\acute{a}^2 + \acute{u})(\acute{b}^2 + \acute{u})(\acute{c}^2 + \acute{u})}}. \tag{2-33}$$

where $c_0$ is an arbitrary constant, and, as exhibited by the equations in (2-30)

$$\acute{\chi} = \begin{cases} \sqrt{\acute{a}^2 + \acute{\xi}} & \text{for } \acute{x}, \\ \sqrt{\acute{b}^2 + \acute{\xi}} & \text{for } \acute{y}, \\ \sqrt{\acute{c}^2 + \acute{\xi}} & \text{for } \acute{z}. \end{cases} \tag{2-34}$$

Thus, by defining[3]

$$\Gamma^x(\acute{\xi}) = \Gamma(\acute{\xi}|\acute{a}^2 + \acute{\xi}), \tag{2-35}$$
$$\Gamma^y(\acute{\xi}) = \Gamma(\acute{\xi}|\acute{b}^2 + \acute{\xi}), \tag{2-36}$$
$$\Gamma^z(\acute{\xi}) = \Gamma(\acute{\xi}|\acute{c}^2 + \acute{\xi}), \tag{2-37}$$

the associated external harmonics which correspond to $\acute{x}$, $\acute{y}$ and $\acute{z}$ are given by the following integrals

$$\begin{aligned} \Gamma^x(\acute{\xi}) &= c_0 \int_{\acute{\xi}}^{\infty} \frac{d\acute{u}}{(\acute{a}^2 + \acute{u})\acute{D}(\acute{u})}, \\ \Gamma^y(\acute{\xi}) &= c_0 \int_{\acute{\xi}}^{\infty} \frac{d\acute{u}}{(\acute{b}^2 + \acute{u})\acute{D}(\acute{u})}, \\ \Gamma^z(\acute{\xi}) &= c_0 \int_{\acute{\xi}}^{\infty} \frac{d\acute{u}}{(\acute{c}^2 + \acute{u})\acute{D}(\acute{u})}. \end{aligned} \tag{2-38}$$

where

$$\acute{D}(\acute{u}) = \sqrt{(\acute{a}^2 + \acute{u})(\acute{b}^2 + \acute{u})(\acute{c}^2 + \acute{u})}. \tag{2-39}$$

It may be observed that the lower integration limit in (2-38) is greater than the zeros of the polynomial in $\acute{D}$, hence there is no singularity within the integration interval.

As we shall immediately see, the constant $c_0$ in the above integrals can be chosen arbitrarily. We may therefore put

$$c_0 = \frac{1}{2}\acute{a}\acute{b}\acute{c}, \tag{2-40}$$

where the factor 1/2 is chosen so as to make[4]

[3] The superscripts on Γ are purely notational.

[4] By logarithmic derivation of the equations in (2-27), $\partial\acute{x}/\partial\acute{\xi} = \frac{1}{2}\acute{x}/(\acute{a}^2 + \acute{\xi})$, $\partial\acute{y}/\partial\acute{\xi} = \frac{1}{2}\acute{y}/(\acute{b}^2 + \acute{\xi})$ and $\partial\acute{z}/\partial\acute{\xi} = \frac{1}{2}\acute{z}/(\acute{c}^2 + \acute{\xi})$. These expressions involve in the relations given by 2-41. Hence we use the factor 1/2 to obtain the unit value.

$$-\left(\acute{x}\frac{\partial\Gamma^x(\acute{\xi})}{\partial\acute{x}}\right)_{\acute{\xi}=0}=-\left(\acute{y}\frac{\partial\Gamma^y(\acute{\xi})}{\partial\acute{y}}\right)_{\acute{\xi}=0}=-\left(\acute{z}\frac{\partial\Gamma^z(\acute{\xi})}{\partial\acute{z}}\right)_{\acute{\xi}=0}=1, \tag{2-41}$$

such that the dimensionless form of the associated external harmonics read

$$\begin{aligned}
\Gamma^x(\acute{\xi}) &= \frac{1}{2}\acute{a}\acute{b}\acute{c}\int_{\acute{\xi}}^{\infty}\frac{d\acute{u}}{(\acute{a}^2+\acute{u})\sqrt{(\acute{a}^2+\acute{u})(\acute{b}^2+u)(\acute{c}^2+\acute{u})}},\\
\Gamma^y(\acute{\xi}) &= \frac{1}{2}\acute{a}\acute{b}\acute{c}\int_{\acute{\xi}}^{\infty}\frac{d\acute{u}}{(\acute{b}^2+\acute{u})\sqrt{(\acute{a}^2+\acute{u})(\acute{b}^2+u)(\acute{c}^2+\acute{u})}},\\
\Gamma^z(\acute{\xi}) &= \frac{1}{2}\acute{a}\acute{b}\acute{c}\int_{\acute{\xi}}^{\infty}\frac{d\acute{u}}{(\acute{c}^2+\acute{u})\sqrt{(\acute{a}^2+\acute{u})(\acute{b}^2+u)(\acute{c}^2+\acute{u})}},
\end{aligned} \tag{2-42}$$

The above choice of $c_0$ is primarily computational in that $\boldsymbol{\Gamma}(\acute{\xi})$ is scaled and relation in (2-41) simplifies the algebraic operations to satisfy the continuity conditions. However, a geometrical interpretation of (2-41) will be given in Chapter 4 when the problem of the composite inclusion is studied.

Finally, by satisfying the boundary condition in (2-21), it follows from (2-26) that

$$\Phi^m(\acute{x},\acute{y},\acute{z}) = -\boldsymbol{j}^T\begin{bmatrix}\acute{x}\\ \acute{y}\\ \acute{z}\end{bmatrix} + \boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{\xi})\begin{bmatrix}\acute{x}\\ \acute{y}\\ \acute{z}\end{bmatrix} \qquad (\acute{x},\acute{y},\acute{z})\in\acute{\Omega}_1, \tag{2-43}$$

where $\boldsymbol{w}$ is constant vector to be determined by the continuity conditions in (2-9) and (2-10), and

$$\boldsymbol{\Gamma}(\acute{\xi}) = \mathrm{diag}(\Gamma^x(\acute{\xi}),\Gamma^y(\acute{\xi}),\Gamma^z(\acute{\xi})). \tag{2-44}$$

Switching to the $(x, y, z)$ coordinate system, a possible analytical solution for distribution of external pressure in $\Omega_1$ in the presence of the boundary condition in (2-11) reads

$$\Phi^m(x,y,z) = \boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{\xi})\boldsymbol{W}\begin{bmatrix}x\\ y\\ z\end{bmatrix} + \Phi_\infty(x,y,z) \qquad (x,y,z)\in\Omega_1, \tag{2-45}$$

where $\boldsymbol{W}$ is the diagonal matrix defined in (2-19).

In the above solution, the term

$$\boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{\xi})\boldsymbol{W}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

describes the fluctuation of pressure in the vicinity of the inclusion. This behaviour is largely due to the integrals in $\boldsymbol{\Gamma}(\acute{\xi})$. Furthermore, since the associated external harmonics vanish at large distances from the surface of the inclusion, the uniform flow at infinity is recovered as the above term approach zero.

### 2.2.4 Associated External Harmonics in Terms of Eccentricities

To make the integration intervals in the associated external harmonics finite, consider the meridian and equatorial eccentricities of an arbitrary ellipsis in

$$\acute{S}_{\acute{\xi}}: \qquad \frac{\acute{x}^2}{\acute{a}^2+\acute{\xi}}+\frac{\acute{y}^2}{\acute{b}^2+\acute{\xi}}+\frac{\acute{z}^2}{\acute{c}^2+\acute{\xi}}=1 \qquad 0\leq\acute{\xi}<\infty. \tag{2-46}$$

Since by definition, the meridian and equatorial eccentricities are the axis ratios of the ellipsoid $\xi$ =cosnt.,

$$\acute{e}=\left(\frac{\acute{c}^2+\acute{\xi}}{\acute{a}^2+\acute{\xi}}\right)^{\frac{1}{2}}; \qquad \acute{E}=\left(\frac{\acute{b}^2+\acute{\xi}}{\acute{a}^2+\acute{\xi}}\right)^{\frac{1}{2}}, \tag{2-47}$$

$\acute{\xi}=0$ gives

$$\acute{e}_0=\frac{\acute{c}}{\acute{a}}=e_0\sqrt{\frac{k_x^m}{k_z^m}}, \qquad \acute{E}_0=\frac{\acute{b}}{\acute{a}}=E_0\sqrt{\frac{k_x^m}{k_y^m}}, \tag{2-48}$$

where $e_0, E_0$ are the eccentricities of the ellipsoid $\xi=0$ whose surface is denoted by $S_0$ (see 2-19). On the other hand, at large distances from $S_0$ (i.e., $\acute{\xi}\to\infty$) the ellipsoids approximate perturbed spheres, and

$$\acute{e}=\acute{E}\to 1: \qquad e_0=\sqrt{\frac{k_z^m}{k_x^m}}, \qquad E_0=\sqrt{\frac{k_y^m}{k_x^m}}. \tag{2-49}$$

Now, by elimination of $\acute{\xi}$ from (2-47) we use (2-48) and obtain

$$\frac{1-\acute{E}^2}{1-\acute{e}^2}=\frac{1-\acute{E}_0^2}{1-\acute{e}_0^2}. \tag{2-50}$$

Noting that

$$\frac{1-\acute{E}_0^2}{1-\acute{e}_0^2}=\frac{\acute{a}^2-\acute{b}^2}{\acute{a}^2-\acute{c}^2}=\acute{\kappa}^2, \tag{2-51}$$

the eccentricities of the ellipsoid are then related by

$$\acute{E}_0^2+\acute{\kappa}^2(1-\acute{e}_0^2)=\acute{E}^2+\acute{\kappa}^2(1-\acute{e}^2)=1, \tag{2-52}$$

Hence by the above relation and the following transformation

$$\acute{\xi}=\acute{a}^2\left(\frac{\acute{e}^2-\acute{e}_0^2}{1-\acute{e}^2}\right), \tag{2-53}$$

the associated external harmonics in terms of the meridian eccentricity are

$$\begin{aligned}
\Gamma^x(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_{\acute{\kappa}}\int_{\acute{e}}^{1}\frac{1-u^2}{\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\,d\acute{u} \\
\Gamma^y(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_{\acute{\kappa}}\int_{\acute{e}}^{1}\frac{1-u^2}{[1-\acute{\kappa}^2(1-u^2)]\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\,d\acute{u} \\
\Gamma^z(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_{\acute{\kappa}}\int_{\acute{e}}^{1}\frac{1-u^2}{\acute{u}^2\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\,d\acute{u}
\end{aligned} \tag{2-54}$$

where,
$$\acute{\Theta}_{\acute{\kappa}} = \frac{\varrho e_0}{(1-\varrho^2 e_0^2)^{\frac{3}{2}}}\sqrt{1-\acute{\kappa}^2(1-\varrho^2 e_0^2)}, \qquad \varrho = \sqrt{k_x^m/k_z^m}. \tag{2-55}$$

Figs. [2-2] and [2-3] depict the profiles of the integrands in (2-54) as well as plots of the associated external harmonics for various values of $\acute{\kappa}^2$.

## 2.3 Computation of External Potential

The computation of the external potential may involve solving the cubic

$$Q(\acute{\alpha}) = \frac{\acute{x}^2}{\acute{a}^2+\acute{\alpha}} + \frac{\acute{y}^2}{\acute{b}^2+\acute{\alpha}} + \frac{\acute{z}^2}{\acute{c}^2+\acute{\alpha}} - 1 = 0, \tag{2-56}$$

for the largest root which we require to be positive[5], and the evaluation of the integrals in (2-42). Solving the cubic numerically will give no difficulty, but straightforward numerical integration of (2-42) or (2-54) may not work well due to, as we will show, the slow convergence of these integrals. Therefore, reduction or normalization of the integrals may be required. Noting that those integrals can be represented as

$$\int R(\acute{u}, f)d\acute{u}, \tag{2-57}$$

where $R$ is a rational function, and $f^2$ is a cubic or quartic polynomial in $\acute{u}$ the expressions for $\Gamma^x, \Gamma^y, \Gamma^z$ are elliptic integrals. Hence, since any elliptic integral can be expressed in terms of the three standard kinds of Legendre-Jacobi elliptic integrals, the computation of (2-42) or (2-54) may probably best handled by reducing the associated external harmonics to elliptic integrals of the first and second kinds which now read

$$F(\acute{\vartheta}|\acute{\kappa}) = \int_0^{\acute{\vartheta}} \frac{d\acute{\phi}}{\sqrt{1-\acute{\kappa}^2\sin^2\acute{\phi}}}, \tag{2-58}$$

and

$$E(\acute{\vartheta}|\acute{\kappa}) = \int_0^{\acute{\vartheta}} \sqrt{1-\acute{\kappa}^2\sin^2\acute{\phi}}\, d\acute{\phi}. \tag{2-59}$$

By the change of variable $\acute{e} = \cos\acute{\vartheta}$, that is,

$$\cos^2\acute{\vartheta} = \frac{\acute{c}^2+\acute{\xi}}{\acute{a}^2+\acute{\xi}} \qquad 0 < \acute{\vartheta} < \pi/2, \tag{2-60}$$

we express $\Gamma^x(\acute{\xi})$, $\Gamma^y(\acute{\xi})$ and $\Gamma^z(\acute{\xi})$ in terms of $F$ and $E$. For instance, by substitution of (2-60) into the expression for $\Gamma^x(\acute{\xi})$, we obtain

$$\Gamma^x(\acute{\vartheta}|\acute{\kappa}) = \frac{\acute{a}\acute{b}\acute{c}}{(\acute{a}^2-\acute{c}^2)^{\frac{3}{2}}} \int_0^{\acute{\vartheta}} \frac{\sin^2\acute{\phi}}{\sqrt{1-\acute{\kappa}^2\sin^2\acute{\phi}}} d\acute{\phi}, \tag{2-61}$$

where $\acute{\kappa}$ is defined in (2-51). Now, noting the identity

[5] Note that for $\acute{\alpha} = 0$, the cubic corresponds to the surface equation of the ellipsoid $\acute{S}_0$

$$\frac{\acute{\kappa}^2 \sin^2 \acute{\phi}}{\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}} = \frac{1}{\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}} - \sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}, \tag{2-62}$$

it follows from (2-61) that

$$\Gamma^x(\acute{\vartheta}|\acute{\kappa}) = \frac{\acute{a}\acute{b}\acute{c}}{(\acute{a}^2 - \acute{c}^2)^{\frac{3}{2}} \acute{\kappa}^2} \left[ F(\acute{\vartheta}|\acute{\kappa}) - E(\acute{\vartheta}|\acute{\kappa}) \right], \tag{2-63}$$

Similarly, $\Gamma^y(\acute{\xi})$ and $\Gamma^z(\acute{\xi})$ may be expressed in terms of standard elliptic integrals. For instance, by exploiting the differentiation of the following reduction formulae[11]

$$\frac{\sin \acute{\phi} \cos \acute{\phi}}{\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}} \quad \text{and} \quad \frac{\sin \acute{\phi} \sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}}{\cos \acute{\phi}}. \tag{2-64}$$

Hence,

$$\Gamma^y(\acute{\vartheta}|\acute{\kappa}) = \frac{\acute{a}\acute{b}\acute{c}}{(\acute{a}^2 - \acute{c}^2)^{\frac{3}{2}} \acute{\kappa}^2 \acute{\kappa}_c^2} \left[ E(\acute{\vartheta}|\acute{\kappa}) - \acute{\kappa}_c^2 F(\acute{\vartheta}|\acute{\kappa}) - \acute{\kappa}^2 \frac{\sin \acute{\vartheta} \cos \acute{\vartheta}}{\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\vartheta}}} \right], \tag{2-65}$$

$$=$$

$$\Gamma^z(\acute{\vartheta}|\acute{\kappa}) = \frac{\acute{a}\acute{b}\acute{c}}{(\acute{a}^2 - \acute{c}^2)^{\frac{3}{2}} \acute{\kappa}_c^2} \left[ \frac{\sin \acute{\vartheta} \sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\vartheta}}}{\cos \acute{\vartheta}} - E(\acute{\vartheta}|\acute{\kappa}) \right]. \tag{2-66}$$

where

$$\acute{\kappa}_c = \sqrt{1 - \acute{\kappa}^2} = \sqrt{\frac{\acute{b}^2 - \acute{c}^2}{\acute{a}^2 - \acute{c}^2}}, \tag{2-67}$$

is called the complementary modulus.

### 2.3.1 Con· ergence

Expanding the integrands in the standard elliptical integrals, it follows from the binomial theorem that

$$\frac{1}{\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}}} = 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m} \acute{\kappa}^{2m} \sin^{2m} \acute{\phi}, \tag{2-68}$$

$$\sqrt{1 - \acute{\kappa}^2 \sin^2 \acute{\phi}} = 1 - \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m} \frac{\acute{\kappa}^{2m} \sin^{2m} \acute{\phi}}{2m-1}. \tag{2-69}$$

Using the reduction formula

$$\int_0^{\pi/2} \sin^{2m} \acute{\phi}\, d\acute{\phi} = \frac{2m-1}{2m} \int_0^{\pi/2} \sin^{2m-2} \acute{\phi}\, d\acute{\phi} = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}, \tag{2-70}$$

the *complete* elliptic integrals (i.e, for which $\acute{\vartheta} = \pi/2$) of the first and second kind are[16, 17]

$$F(\pi/2|\acute{\kappa}) = \frac{\pi}{2}\left[1+\sum_{m=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\right)^2 \acute{\kappa}^{2m}\right], \tag{2-71}$$

$$E(\pi/2|\acute{\kappa}) = \frac{\pi}{2}\left[1-\sum_{m=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\right)^2 \frac{\acute{\kappa}^{2m}}{2m-1}\right]. \tag{2-72}$$

The above series converge absolutely and uniformly for $|\acute{\kappa}^2| < 1$. Hence it follows from (2-51) that the associated external harmonics converge for

$$\acute{e}_0^2 < \acute{E}_0^2 < 1 \qquad \text{i.e,} \quad e_0 < E_0\sqrt{\frac{k_z^m}{k_y^m}} < \sqrt{\frac{k_z^m}{k_x^m}}. \tag{2-73}$$

By the preceding series, the associated external harmonic $\Gamma^x$ may be written as

$$\begin{aligned}\sigma\Gamma^x(\pi/2|\acute{\kappa}) = 1 &+ \frac{8}{3}\frac{1^2\,3^2}{2^2\,4^2}\acute{\kappa}^2 + \frac{12}{5}\frac{1^2\,3^2\,5^2}{2^2\,4^2\,6^2}\acute{\kappa}^4 \\ &+ \frac{16}{7}\frac{1^2\,3^2\,5^2\,7^2}{2^2\,4^2\,6^2\,8^2}\acute{\kappa}^6 + \frac{20}{9}\frac{1^2\,3^2\,5^2\,7^2\,9^2}{2^2\,4^2\,6^2\,8^2\,10^2}\acute{\kappa}^8 + \cdots\cdots.\end{aligned} \tag{2-74}$$

Here

$$\frac{1}{\sigma} = \frac{\pi}{4}\frac{\acute{a}\acute{b}\acute{c}}{(\acute{a}^2-\acute{c}^2)^{3/2}}. \tag{2-75}$$

Obviously, the above series converges slowly for $|\acute{\kappa}^2| < 1$.

### 2.3.2 Analyticity

For convenience, we may consider the elliptic function (i.e, the inverse of the elliptic integral) of the first kind. By the change of variable, $\acute{t} = \sin\phi$, the elliptic integral of 1st kind takes on the form

$$F(T|\acute{\kappa}) = \int_0^{\acute{T}} \frac{d\acute{t}}{\sqrt{(1-\acute{t}^2)(1-\acute{\kappa}^2 t^2)}}. \tag{2-76}$$

Differentiating the above integral, we then take the square of each side and obtain the following non-linear differential equation

$$\left(\frac{d\acute{T}}{dF}\right)^2 = (1-\acute{T}^2)(1-\acute{\kappa}^2 T^2), \tag{2-77}$$

which obviously admits the following analytical cases

$$\left(\frac{d\acute{T}}{dF}\right)^2 = \begin{cases} 1-\acute{T}^2 & , \quad \acute{\kappa}^2 = 0, \\ (1-\acute{T}^2)^2 & , \quad \acute{\kappa}^2 = 1. \end{cases} \tag{2-78}$$

Solving the equations, we get

$$F(T|\acute{\kappa}=0) = \sin^{-1} T, \qquad F(T|\acute{\kappa}=1) = \frac{1}{2}\ln\frac{1+T}{1-T}. \tag{2-79}$$

Hence the associated external harmonics have trigonometric and hyperbolic analytical solutions when $\acute{\kappa}^2 = 0$ and $\acute{\kappa}^2 = 1$, respectively. In geometrical context, the above cases signify the deformation of the ellipsoid whose surface is $\acute{S}_0$, to the spheroids $\acute{a} = \acute{b} > \acute{c}$ (for $\acute{\kappa}^2 = 0$) and $\acute{b} = \acute{c} < \acute{a}$ (for $\acute{\kappa}^2 = 1$).

### 2.3.3 Analytical E aluation of Associated External Harmonics

Apart from preceding cases in which $\acute{\kappa}^2 = 0, 1$, there are no primitive functions for the integrands in (2-54). Therefore, $\Gamma^x$, $\Gamma^y$ and $\Gamma^z$ are essentially to be evaluated numerically. For the analytical case $\acute{\kappa}^2 = 0$,

$$
\begin{aligned}
\Gamma^x(\acute{e}|0) = \Gamma^y(\acute{e}|0) &= \acute{\Theta}_0 \int_{\acute{e}}^{1} \frac{1-u^2}{u\sqrt{1-u^2}}\, du = \acute{\Theta}_0 \int_{\acute{e}}^{1} \left( \frac{1}{u\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} \right) du, \\
&= \acute{\Theta}_0 \int_{\acute{e}}^{1} \frac{1}{u\sqrt{1-u^2}}\, du - \acute{\Theta}_0 \int_{\acute{e}}^{1} \frac{\acute{e}}{\sqrt{1-u^2}}\, du, \\
&= \frac{1}{2}\acute{\Theta}_0 \left[ \tan^{-1}\left(\frac{u}{\sqrt{1-u^2}}\right) + u\sqrt{1-u^2} \right]_{\acute{e}}^{1}, \\
&= \frac{1}{2}\acute{\Theta}_0 \left[ \tan^{-1}\left(\frac{\sqrt{1-\acute{e}^2}}{\acute{e}}\right) - \acute{e}\sqrt{1-\acute{e}^2} \right],
\end{aligned}
\tag{2-80}
$$

and

$$
\begin{aligned}
\Gamma^z(\acute{e}|0) &= \acute{\Theta}_0 \int_{\acute{e}}^{1} \frac{1-u^2}{\acute{u}^2\sqrt{1-u^2}}\, du, \\
&= \acute{\Theta}_0 \left[ \frac{\sqrt{1-u^2}}{u} - \tan^{-1}\left(\frac{u}{\sqrt{1-u^2}}\right) \right]_{\acute{e}}^{1}, \\
&= \acute{\Theta}_0 \left[ \frac{\sqrt{1-\acute{e}^2}}{\acute{e}} - \tan^{-1}\left(\frac{\sqrt{1-\acute{e}^2}}{\acute{e}}\right) \right].
\end{aligned}
\tag{2-81}
$$

where

$$
\acute{\Theta}_0 = \frac{\varrho e_0}{(1-\varrho^2 e_0^2)^{\frac{3}{2}}}.
\tag{2-82}
$$

In the other analytical case, $\acute{\kappa}^2 = 1$

$$
\begin{aligned}
\Gamma^x(\acute{e}|1) &= \acute{\Theta}_1 \int_{\acute{e}}^{1} \frac{1-u^2}{u\sqrt{1-u^2}}\, du = \acute{\Theta}_1 \int_{\acute{e}}^{1} \left( \frac{1}{u\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} \right) du \\
&= \frac{1}{2}\acute{\Theta}_1 \int_{\acute{e}}^{1} \left( \frac{1}{u\sqrt{1-u^2}} - \frac{\sqrt{1-u^2}}{u} \right) du - \acute{\Theta}_1 \int_{\acute{e}}^{1} \frac{u}{\sqrt{1-u^2}}\, du \\
&= \frac{1}{2}\acute{\Theta}_1 \left[ \ln\frac{1+\sqrt{1-u^2}}{1-\sqrt{1-u^2}} \right]_{1}^{\acute{e}} - \left[ \sqrt{1-u^2} \right]_{1}^{\acute{e}}, \\
&= \acute{\Theta}_1 \left( \frac{1}{2}\ln\frac{1+\sqrt{1-\acute{e}^2}}{1-\sqrt{1-\acute{e}^2}} - \sqrt{1-\acute{e}^2} \right).
\end{aligned}
\tag{2-83}
$$

where

$$\acute{\Theta}_1 = \frac{\varrho^2 e_0^2}{(1-\varrho^2 e_0^2)^{\frac{3}{2}}}. \tag{2-84}$$

and

$$\Gamma^y(\acute{e}|1) = \Gamma^z(\acute{e}|1) = \acute{\Theta}_1 \int_{\acute{e}}^{1} \frac{1-u^2}{u^3\sqrt{1-u^2}}\, du, \tag{2-85}$$

which, by the trigonometric substitution $u = \sin\acute{\vartheta}$,

$$\begin{aligned}
\Gamma^y(\acute{e}|1) = \Gamma^z(\acute{e}|1) &= \acute{\Theta}_1 \int_{\arcsin\acute{e}}^{\frac{\pi}{2}} \frac{d\acute{\vartheta}}{\sin^3\acute{\vartheta}}, \\
&= \frac{1}{2}\acute{\Theta}_1 \left[\cot\acute{\vartheta}\csc\acute{\vartheta} + \ln\frac{\sin\acute{\vartheta}/2}{\cos\acute{\vartheta}/2}\right]_{\arcsin\acute{e}}^{\frac{\pi}{2}}, \\
&= \frac{1}{2}\acute{\Theta}_1 \left[\frac{\sqrt{1-\sin^2\acute{\vartheta}}}{\sin^2\acute{\vartheta}} - \frac{1}{2}\ln\frac{1+\sqrt{1-\sin^2\acute{\vartheta}}}{1-\sqrt{1+\sin^2\acute{\vartheta}}}\right]_{\arcsin\acute{e}}^{\frac{\pi}{2}}, \\
&= \frac{1}{2}\acute{\Theta}_1 \left(\frac{\sqrt{1-\acute{e}^2}}{\acute{e}^2} - \frac{1}{2}\ln\frac{1+\sqrt{1-\acute{e}^2}}{1-\sqrt{1-\acute{e}^2}}\right).
\end{aligned} \tag{2-86}$$

**Surface Value of Associated External Harmonics**

Let

$$v(\acute{e}) = \frac{(1-\acute{e}^2)^{\frac{3}{2}}/\acute{e}}{\acute{\Theta}_0} = \frac{(1-\acute{e}^2)^{\frac{3}{2}}/\acute{e}}{(1-\acute{e}_0^2)^{\frac{3}{2}}/\acute{e}_0}, \tag{2-87}$$

such that, for $\acute{\kappa}^2 = 0$

$$\Gamma^x(\acute{e}|0) = \Gamma^y(\acute{e}|0) = \frac{1}{2}\lambda(\acute{e}|0)v(\acute{e}), \qquad \Gamma^z(\acute{e}|0) = [1-\lambda(\acute{e}|0)]v(\acute{e}), \tag{2-88}$$

where we defined

$$\lambda(\acute{e}|0) = \frac{\acute{e}^2}{1-\acute{e}^2}\left[\frac{1}{\acute{e}\sqrt{1-\acute{e}^2}}\tan^{-1}(\frac{\sqrt{1-\acute{e}^2}}{\acute{e}}) - 1\right]. \tag{2-89}$$

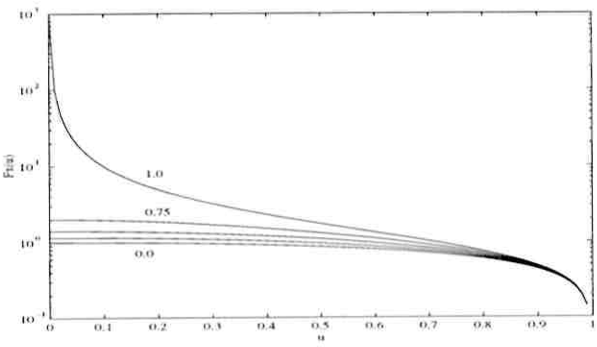
Accordingly, we find that

$$\Gamma^x(\acute{e}|0) + \Gamma^y(\acute{e}|0) + \Gamma^z(\acute{e}|0) = v(\acute{e}). \tag{2-90}$$

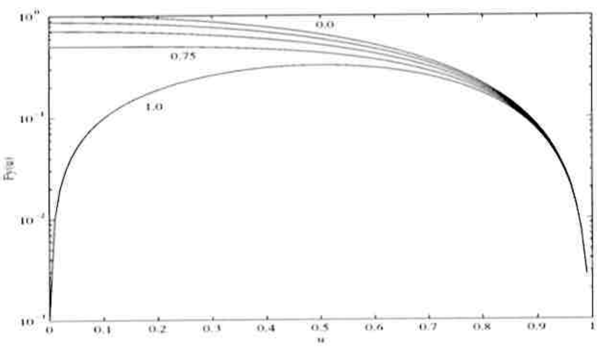
Obviously, when the distances from $\acute{S}_0$ approach infinity i.e., ($\acute{e} \to 1$), then $v \to 0$, and the associated external harmonics vanish. On the other hand, if the above relation is evaluated on the surface $\acute{S}_0$, we obtain

$$\Gamma^x(\varrho e_0|\acute{\kappa}) + \Gamma^y(\varrho e_0|\acute{\kappa}) + \Gamma^z(\varrho e_0|\acute{\kappa}) = 1. \tag{2-91}$$

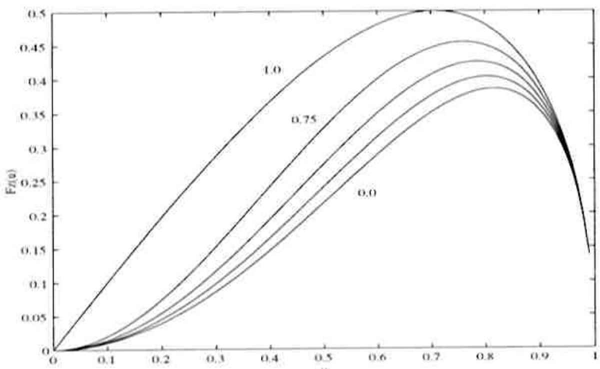
This relation holds for general ellipsoids[7].




($\boldsymbol{A}$) Integrand of $\Gamma^x(\acute{e}|\acute{\kappa})$ vs. $\acute{e}$ at various values of $\acute{\kappa}^2$




($\boldsymbol{B}$) Integrand of $\Gamma^y(\acute{e}|\acute{\kappa})$ vs. $\acute{e}$ at various values of $\acute{\kappa}^2$




($\boldsymbol{C}$) Integrand of $\Gamma^z(\acute{e}|\acute{\kappa})$ vs. $\acute{e}$ at various values of $\acute{\kappa}^2$

Figure 2-2: *The integrands of the associated external harmonics as function of meridian eccentricity for* $\acute{\kappa}^2 = 0, 0.25, 0.5, 0.75, 1$

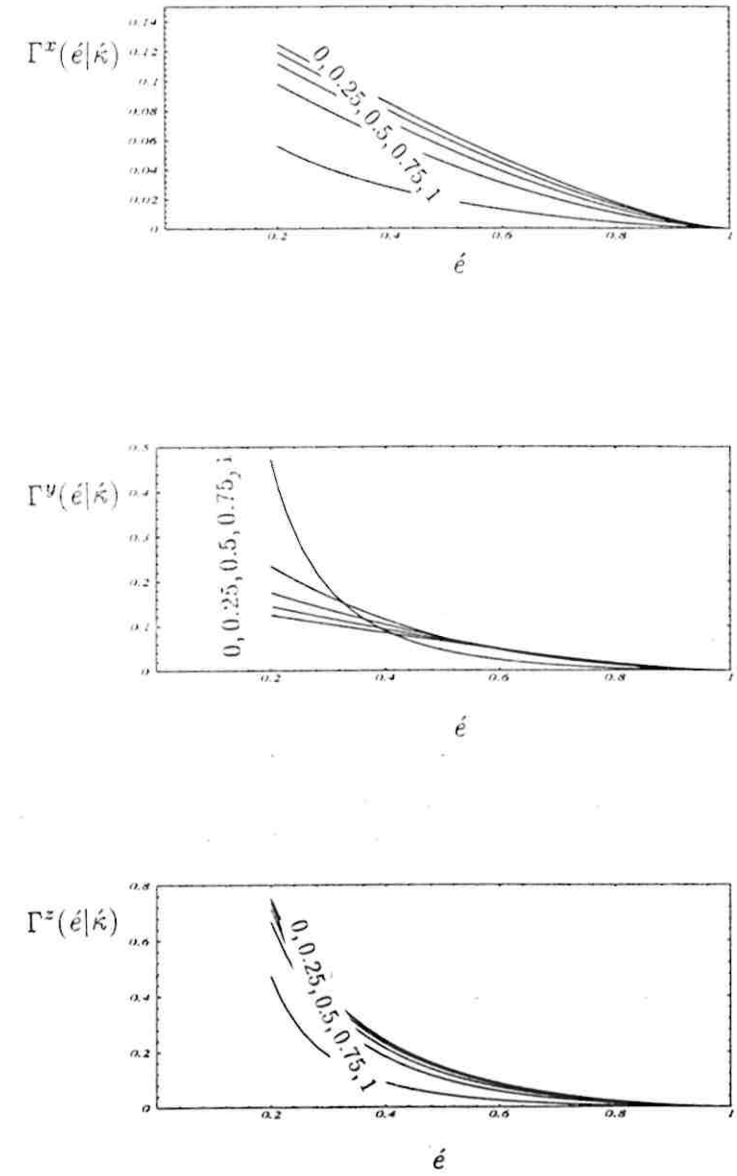



Figure 2-3: *Associated external harmonics as function of meridian eccentricity for* $\acute{\kappa}^2 = 0, 0.25, 0.5, 0.75, 1$

## 2.4 Continuity Conditions

We are now in a position to apply the continuity conditions in (2-9) and (2-10) in order to determine the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in the solutions[6] given by (2-18) and (2-45). Since it is required that the potential and the normal flux must be continuous everywhere on $S_0$, we may simplify the subsequent algebraic operations by introducing parametrization of the surface points. For this purpose, let $c_0 \subseteq R^2$ be the region

$$c_0 = \{(\mu, \eta) \in R^2 | -1 \leq \mu \leq 1, \quad -1 \leq \eta \leq 1\}. \tag{2-92}$$

and consider the parametrization

$$z = c\mu, \qquad y = b\sqrt{(1-\mu^2)(1-\eta)}, \qquad x = a\eta\sqrt{1-\mu^2}. \tag{2-93}$$

Then $|\mu| = 1$ corresponds to the pole points $(0, 0, \pm c)$ and $|\mu| < 1$ to the equatorial ellipses,

$$\frac{x^2}{a(1-\mu^2)} + \frac{y^2}{b^2(1-\mu^2)} = 1 \qquad (x, y) \in S_0; \quad |\mu| < 1. \tag{2-94}$$

Writing (2-93) in matrix form, a parametrization of surface points $(x, y, z) \in S_0$ is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{C}\boldsymbol{\omega}(\mu, \eta) \qquad (x, y, z) \in S_0; \quad (\mu, \eta) \in c_0, \qquad \boldsymbol{C} = \mathrm{diag}(a, b, c), \tag{2-95}$$

where we defined

$$\boldsymbol{\omega}(\mu, \eta) = \begin{bmatrix} \eta\sqrt{1-\mu^2} \\ \sqrt{(1-\eta)(1-\mu^2)} \\ \mu \end{bmatrix} \qquad |\eta| \leq 1; \quad |\mu| \leq 1, \tag{2-96}$$

### 2.4.1 Continuity of Potential

Equating (2-45) and (2-18), the condition in (2-9) reads

$$\boldsymbol{v}^T\mathbf{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (\boldsymbol{w}^T\boldsymbol{\Gamma}(\varrho e_0)\boldsymbol{W} - \boldsymbol{J}^T) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_0 \tag{2-97}$$

Using the parametrization in (2-95), the above equation becomes

$$(\boldsymbol{v}^T\mathbf{V} - \boldsymbol{w}^T\boldsymbol{\Gamma}(\varrho e_0)\boldsymbol{W} + \boldsymbol{J}^T)\boldsymbol{C}\boldsymbol{\omega}(\mu, \eta) = 0 \qquad |\mu| \leq 1, \quad |\mu| \leq 1. \tag{2-98}$$

To satisfy this equation for $|\mu| \leq 1$ and $|\eta| \leq 1$, we must require

$$(\boldsymbol{v}^T\mathbf{V} - \boldsymbol{w}^T\boldsymbol{\Gamma}(\varrho e_0)\boldsymbol{W} + \boldsymbol{J}^T)\boldsymbol{C} = \mathbf{0}. \tag{2-99}$$

[6] For notational simplicity, we proceed henceforth without explicitly giving $\acute{\kappa}$ in the elliptic integrals $\Gamma^x, \Gamma^y, \Gamma^z$

### 2.4.2 Continuity of the Normal Flux

In reference to (2-1), the matrix form of the unit vector normal to $S_0$ may be written as

$$\boldsymbol{n}(x,y,z) = \varepsilon \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{where} \quad \frac{2}{\varepsilon} = \|\nabla( \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1)\| \quad (x,y,z) \in S_0, \quad (2\text{-}100)$$

Furthermore, we designate by $f_P^+$ and $f_P^-$ the right- and left-hand side of (2-10) i.e.,

$$f_P^+ = k_x^m \frac{\partial \Phi^m}{\partial x} n_x + k_y^m \frac{\partial \Phi^m}{\partial y} n_y + k_z^m \frac{\partial \Phi^m}{\partial z} n_z \quad (\text{on } S_0), \quad (2\text{-}101)$$

$$f_P^- = k_x^i \frac{\partial \Phi^i}{\partial x} n_x + k_y^i \frac{\partial \Phi^i}{\partial y} n_y + k_z^i \frac{\partial \Phi^i}{\partial z} n_z \quad (\text{on } S_0). \quad (2\text{-}102)$$

Since we assume linearity of potential in $\boldsymbol{J}$, we may derive expressions for $f_P^+$ and $f_P^-$ by considering the separate solutions for the pressure distributions. For instance,

$$\begin{aligned} k_x^m \frac{\partial \Phi^m}{\partial x} n_x &= k_x^m \frac{\partial}{\partial x} \left[ -J_x x + A \sqrt{\frac{k^m}{k_x^m}} x \Gamma^x(\acute{\xi}) \right], \\ &= -J_x k_x^m n_x + A k_x^m \sqrt{\frac{k^m}{k_x^m}} \frac{\partial}{\partial \acute{x}} \left[ \acute{x} \Gamma^x(\acute{\xi}) \right] n_x, \\ &= -J_x k_x^m n_x + A k_x^m \sqrt{\frac{k^m}{k_x^m}} \left[ \Gamma^x(\acute{\xi}) + \acute{x} \frac{\partial}{\partial \acute{x}} \Gamma^x(\acute{\xi}) \right] n_x. \end{aligned} \quad (2\text{-}103)$$

Evaluating the above equation on $S_0$, we use the relations in (2-41) and obtain

$$\left( k_x^m \frac{\partial \Phi^m}{\partial x} n_x \right)_{S_0} = -\varepsilon J_x k_x^m \frac{x}{a^2} + \varepsilon A k_x^m \sqrt{\frac{k^m}{k_x^m}} (\Gamma^x(\varrho e_0) - 1) \frac{x}{a^2}. \quad (2\text{-}104)$$

Hence, by symmetry,

$$\left( k_y^m \frac{\partial \Phi^m}{\partial y} n_y \right)_{S_0} = -\varepsilon J_y k_y^m \frac{y}{b^2} + \varepsilon B k_y^m \sqrt{\frac{k^m}{k_y^m}} (\Gamma^y(\varrho e_0) - 1) \frac{y}{b^2}. \quad (2\text{-}105)$$

$$\left( k_z^m \frac{\partial \Phi^m}{\partial z} n_z \right)_{S_0} = -\varepsilon J_z k_z^m \frac{z}{c^2} + \varepsilon C k_z^m \sqrt{\frac{k^m}{k_z^m}} (\Gamma^z(\varrho e_0) - 1) \frac{z}{c^2}. \quad (2\text{-}106)$$

Thus,

$$f_P^+ = \varepsilon (\boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \boldsymbol{W} - \boldsymbol{J}^T) \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (x,y,z) \in S_0. \quad (2\text{-}107)$$

By similar procedure, we obtain

$$f_{\bar{P}} = \varepsilon \boldsymbol{v}^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_0. \tag{2-108}$$

Equating (2-107) and (2-108), the condition in (2-10) reads

$$\boldsymbol{v}^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (\boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \boldsymbol{W} - \boldsymbol{J}^T) \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_0. \tag{2-109}$$

Furthermore, by utilization of (2-95),

$$(\boldsymbol{v}^T \boldsymbol{k}^i \boldsymbol{V} - \boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \boldsymbol{W} + \boldsymbol{J}^T) \boldsymbol{D} \boldsymbol{C} \boldsymbol{\omega}(\mu, \eta) = 0 \qquad |\mu| \leq 1, \quad |\mu| \leq 1. \tag{2-110}$$

For $|\mu| \leq 1$ and $|\eta| \leq 1$, the above equation is satisfied by requiring

$$(\boldsymbol{v}^T \boldsymbol{k}^i \boldsymbol{V} - \boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \boldsymbol{W} + \boldsymbol{J}^T) \boldsymbol{D} \boldsymbol{C} = \boldsymbol{0}. \tag{2-111}$$

We now multiply by the equations in (2-99) and (2-111) by $\boldsymbol{C}^{-1}$ and $\boldsymbol{C}^{-1}\boldsymbol{D}^{-1}$, respectively, from the right-hand side and obtain the following linear system of equations

$$\begin{bmatrix} \boldsymbol{V} & -\boldsymbol{W}\boldsymbol{\Gamma}(\varrho e_0) \\ \boldsymbol{k}^i \boldsymbol{V} & -\boldsymbol{k}^m \boldsymbol{W}(\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{J} \\ -\boldsymbol{k}\boldsymbol{J} \end{bmatrix}, \tag{2-112}$$

which has precisely one solution if

$$\begin{aligned} \text{Det} \begin{bmatrix} \boldsymbol{V} & -\boldsymbol{W}\boldsymbol{\Gamma}(\varrho e_0) \\ \boldsymbol{k}^i \boldsymbol{V} & -\boldsymbol{k}^m \boldsymbol{W}(\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \end{bmatrix} &= -\boldsymbol{V}\boldsymbol{W}((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m), \\ &= -\boldsymbol{V}\boldsymbol{W}\boldsymbol{k}^m((\boldsymbol{I} - (\boldsymbol{k}^m)^{-1}\boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{I}) \neq 0 \end{aligned} \tag{2-113}$$

Thus, provided that

$$\boldsymbol{k}^m > \boldsymbol{0}; \qquad \boldsymbol{k}^i > \boldsymbol{0}, \tag{2-114}$$

the matrix in (2-112) is singular if

$$(\boldsymbol{I} - (\boldsymbol{k}^m)^{-1}\boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) = \boldsymbol{I}, \tag{2-115}$$

we must therefore have

$$(\boldsymbol{k}^m)^{-1}\boldsymbol{k}^i \neq \boldsymbol{I} - \boldsymbol{\Gamma}^{-1}(\varrho e_0). \tag{2-116}$$

Assuming (2-114) and (2-116) are satisfied, we solve (2-112) for $\boldsymbol{v}$ and $\boldsymbol{w}$, and obtain

$$\boldsymbol{v}^T = \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}\boldsymbol{k}^m\boldsymbol{V}^{-1}, \tag{2-117}$$
$$\boldsymbol{w}^T = \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{W}^{-1}. \tag{2-118}$$

Thus, the distribution of pressure inside and outside the surface of the inclusion read

$$\Phi^i(x,y,z) = \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}\boldsymbol{k}^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{in } \Omega_0, \tag{2-119}$$

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\acute{e}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{in } \Omega_1. \tag{2-120}$$

It may be observed that the vectors $\boldsymbol{v}^T = (A^i_x, A^i_y, A^i_z)$ and $\boldsymbol{w}^T = (A_x, A_y, A_z)$ can be written in component form as follows,

$$A^i_\lambda = J_\lambda \frac{k^m_\lambda}{(k^m_\lambda - k^i_\lambda)\Gamma^\lambda(\varrho e_0) - k^m_\lambda}\sqrt{\frac{k^i_\lambda}{k^i}}, \qquad \lambda = x, y, z. \tag{2-121}$$

$$A_\lambda = J_\lambda \frac{k^m_\lambda - k^i_\lambda}{(k^m_\lambda - k^i_\lambda)\Gamma^\lambda(\varrho e_0) - k^m_\lambda}\sqrt{\frac{k^m_\lambda}{k^m}}, \qquad \lambda = x, y, z. \tag{2-122}$$

## 2.5 Discussion on the Generalized Inclusion Model

### 2.5.1 Verification of Solutions

We summerize now the inclusion problem solved in this chapter by first verifying the developed analytical solutions. We will then make a brief discussion on specific cases of the generalized inclusion model presented here. Finally, we show that analytical solutions reported in previous works are special cases of the generalized solution developed in this study. of similar character.

- **Uniform Flow at Infinity**: At large distances from the inclusion, $\acute{e} \to 1$ as the ellipsoids approximate spheres. Thus, by noting that the associated external harmonics vanish at infinity, it follows from (2-120) that

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{as} \quad \|(x,y,z)\| \to \infty. \tag{2-123}$$

  as it should do. Hence the boundary condition in (2-11) is recovered.

- **Continuity of the potential**: On the surface of the inclusion, the matrix $\boldsymbol{\Gamma}(\acute{e})$ in the external solution takes on its surface value $\boldsymbol{\Gamma}(\acute{e}_0)$. Thus, the distribution of the external pressure reads

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} +$$

$$+ \quad \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\acute{e}) \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$= \quad -\boldsymbol{J}^T \left(\boldsymbol{I} - ((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0)\right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2\text{-}124)$$

Noting the identity

$$\boldsymbol{I} = ((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m), \quad (2\text{-}125)$$

the equation may be written as

$$\Phi^m(x, y, z) = \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}\boldsymbol{k}^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (\text{on } S_0). \quad (2\text{-}126)$$

Comparing this equation to (2-119) shows

$$\Phi^m(x, y, z) = \Phi^i(x, y, z) = \boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}\boldsymbol{k}^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (\text{on } S_0). \quad (2\text{-}127)$$

Hence the solutions in (2-119) and (2-120) satisfy the continuity condition in (2-9).

- **Continuity of the normal flux:** The continuity condition in (2-10) requires that $f_P^+ = f_P^-$, where $f_P^+$ and $f_P^-$ are defined in (2-101) and (2-102). Let $P(x, y, z) \in S_0$ be an arbitrary surface point and consider the distribution for the external pressure in (2-45). By the chain rule, we may write

$$f_P^+ = (k_x^m \frac{\partial \Phi^m}{\partial \acute{\xi}} \frac{\partial \acute{\xi}}{\partial \acute{x}} \frac{\partial \acute{x}}{\partial x} n_x + k_y^m \frac{\partial \Phi^m}{\partial \acute{\xi}} \frac{\partial \acute{\xi}}{\partial \acute{y}} \frac{\partial \acute{x}}{\partial y} n_y + k_z^m \frac{\partial \Phi^m}{\partial \acute{\xi}} \frac{\partial \acute{\xi}}{\partial \acute{z}} \frac{\partial \acute{z}}{\partial x} n_z)_{S_0}, \quad (2\text{-}128)$$

where

$$\left(\frac{\partial \acute{x}}{\partial \acute{\xi}}\right)_{\acute{\xi}=0} = \frac{\acute{x}}{2\acute{a}^2}, \qquad \left(\frac{\partial \acute{y}}{\partial \acute{\xi}}\right)_{\acute{\xi}=0} = \frac{\acute{y}}{2\acute{b}^2}, \qquad \left(\frac{\partial \acute{z}}{\partial \acute{\xi}}\right)_{\acute{\xi}=0} = \frac{\acute{z}}{2\acute{c}^2}. \quad (2\text{-}129)$$

Recalling that $k^m = (k_x^m k_y^m k_z^m)^{1/3}$, we substitute the above expressions into (2-128) and find

$$\begin{aligned} f_P^+ &= 2\varepsilon(k_x^m \sqrt{\frac{k^m}{k_x^m}} \frac{x/a_2}{\acute{x}/\acute{a}^2} + k_y^m \sqrt{\frac{k^m}{k_y^m}} \frac{y/b^2}{\acute{y}/\acute{b}^2} + k_z^m \sqrt{\frac{k^m}{k_z^m}} \frac{z/c^2}{\acute{z}/\acute{c}^2}) \left(\frac{\partial \Phi^m}{\partial \acute{\xi}}\right)_{\acute{\xi}=0}, \\ &= 6\varepsilon k^m \left(\frac{\partial \Phi^m}{\partial \acute{\xi}}\right)_{\acute{\xi}=0} \qquad \text{where} \quad k^m = (k_x^m k_y^m k_z^m)^{1/3} \end{aligned} \quad (2\text{-}130)$$

where

$$\left(\frac{\partial\Phi^m}{\partial\acute{\xi}}\right)_{\acute{\xi}=0} = -\frac{\partial}{\partial\acute{\xi}}(\acute{\boldsymbol{J}}^T\begin{bmatrix}\acute{x}\\\acute{y}\\\acute{z}\end{bmatrix} - \boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{\xi})\begin{bmatrix}\acute{x}\\\acute{y}\\\acute{z}\end{bmatrix})_{\acute{\xi}=0},$$

$$= -\{\acute{\boldsymbol{J}}^T\begin{bmatrix}\partial\acute{x}/\partial\acute{\xi}\\\partial\acute{y}/\partial\acute{\xi}\\\partial\acute{z}/\partial\acute{\xi}\end{bmatrix} - \boldsymbol{w}^T(\frac{\partial\boldsymbol{\Gamma}}{\partial\acute{\xi}}\begin{bmatrix}\acute{x}\\\acute{y}\\\acute{z}\end{bmatrix} + \boldsymbol{\Gamma}\begin{bmatrix}\partial\acute{x}/\partial\acute{\xi}\\\partial\acute{y}/\partial\acute{\xi}\\\partial\acute{z}/\partial\acute{\xi}\end{bmatrix})\}_{\acute{\xi}=0},$$

$$= -\frac{1}{2}\acute{\boldsymbol{J}}^T\begin{bmatrix}\acute{x}/\acute{a}^2\\\acute{y}/\acute{b}^2\\\acute{z}/\acute{c}^2\end{bmatrix} - \frac{1}{2}\boldsymbol{w}^T(\boldsymbol{\Gamma}(0)-\boldsymbol{I})\begin{bmatrix}\acute{x}/\acute{a}^2\\\acute{y}/\acute{b}^2\\\acute{z}/\acute{c}^2\end{bmatrix}_{P\in S_0},$$

$$= -\frac{1}{2}(\acute{\boldsymbol{J}}^T - \boldsymbol{w}^T(\boldsymbol{\Gamma}(0)-\boldsymbol{I}))\acute{\boldsymbol{D}}\begin{bmatrix}\acute{x}\\\acute{y}\\\acute{z}\end{bmatrix}_{P\in S_0} \qquad (2\text{-}131)$$

where $\boldsymbol{\Gamma}(0)$ expresses the value of the $\boldsymbol{\Gamma}$ on the surface of the inclusion. Thus,

$$f_P^+ = -3\varepsilon k^m(\acute{\boldsymbol{J}}^T - \boldsymbol{w}^T(\boldsymbol{\Gamma}(0)-\boldsymbol{I}))\acute{\boldsymbol{D}}\begin{bmatrix}\acute{x}\\\acute{y}\\\acute{z}\end{bmatrix}_{P\in S_0}. \qquad (2\text{-}132)$$

Hence by the transformations in (2-19, 2-22, 2-24),

$$f_P^+ = -3\varepsilon k^m(\boldsymbol{J}^T - \boldsymbol{w}^T(\boldsymbol{\Gamma}(0)-\boldsymbol{I})\boldsymbol{W})\boldsymbol{W}^{-2}\boldsymbol{D}\begin{bmatrix}x\\y\\z\end{bmatrix}_{P\in S_0}. \qquad (2\text{-}133)$$

Furthermore, by noting that

$$k^m\boldsymbol{W}^{-2} = k^m\,\mathrm{diag}(\frac{k_x^m}{k^m}, \frac{k_y^m}{k^m}, \frac{k_z^m}{k^m}) = \boldsymbol{k}^m, \qquad (2\text{-}134)$$

we substitute (2-118) into (2-133) and obtain

$$f_P^+ = -3\varepsilon\boldsymbol{J}^T(\boldsymbol{I} - ((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i))\boldsymbol{k}^m\boldsymbol{D}\begin{bmatrix}x\\y\\z\end{bmatrix}_{P\in S_0}. \qquad (2\text{-}135)$$

Finally, by the identity,

$$\boldsymbol{I} = ((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m), \qquad (2\text{-}136)$$

we obtain

$$f_P^+ = 3\varepsilon\boldsymbol{J}^T((\boldsymbol{k}^m - \boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - \boldsymbol{k}^i))\boldsymbol{k}^i\boldsymbol{k}^m\boldsymbol{D}\begin{bmatrix}x\\y\\z\end{bmatrix}_{P\in S_0}. \qquad (2\text{-}137)$$

Similarly, consider the distribution of the internal pressure in (2-18). Similarly, we derive an expression for $f_P^-$ by first employing the chain rule

$$f_P^- = \left[k_x^i \frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial x}n_x + k_y^i \frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{y}}\frac{\partial\tilde{x}}{\partial y}n_y + k_z^i \frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{z}}\frac{\partial\tilde{z}}{\partial x}n_z\right]_{P\in S_0}, \tag{2-138}$$

where, by implicit derivation of the surface equation in (2-15),

$$\frac{\partial\tilde{x}}{\partial\tilde{\lambda}} = \tilde{x}\frac{\tilde{\lambda}}{\tilde{\lambda}^2+\tilde{p}}, \qquad \frac{\partial\tilde{y}}{\partial\tilde{\lambda}} = \tilde{y}\frac{\tilde{\lambda}}{\tilde{\lambda}^2+\tilde{q}}, \qquad \frac{\partial\tilde{z}}{\partial\tilde{\lambda}} = \tilde{z}\frac{1}{\tilde{\lambda}}. \tag{2-139}$$

Substitution of the above in the preceding equation gives

$$\begin{aligned}
f_P^- &= \varepsilon k^i \left(\frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{x}}\frac{\tilde{x}}{\tilde{a}^2} + \frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{y}}\frac{\tilde{y}}{\tilde{a}^2} + \frac{\partial\Phi^i}{\partial\tilde{\lambda}}\frac{\partial\tilde{\lambda}}{\partial\tilde{z}}\frac{\tilde{z}}{\tilde{a}^2}\right)_{\tilde{\lambda}=\tilde{c}}, \\
&= 3\varepsilon\frac{k^i}{\tilde{c}}\left(\frac{\partial\Phi^i}{\partial\tilde{\lambda}}\right)_{\tilde{\lambda}=\tilde{c}} = 3\varepsilon\frac{k^i}{\tilde{c}}\boldsymbol{v}^T\begin{bmatrix}\partial\tilde{x}/\partial\tilde{\lambda}\\ \partial\tilde{y}/\partial\tilde{\lambda}\\ \partial\tilde{z}/\partial\tilde{\lambda}\end{bmatrix}_{\tilde{\lambda}=\tilde{c}}, \\
&= 3\varepsilon\frac{k^i}{\tilde{c}}\begin{bmatrix}\tilde{\lambda}/(\tilde{\lambda}^2+\tilde{p}) & & \\ & \tilde{\lambda}/(\tilde{\lambda}^2+\tilde{q}) & \\ & & 1/\tilde{\lambda}\end{bmatrix}_{\tilde{\lambda}=\tilde{c}}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix}_{P\in S_0}, \\
&= 3\varepsilon k^i\tilde{\boldsymbol{D}}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix}_{P\in S_0}.
\end{aligned} \tag{2-140}$$

Referring to the transformations defined in (2-13, 2-16), we obtain

$$f_P^- = 3\varepsilon k^i\boldsymbol{v}^T\boldsymbol{V}^{-2}\boldsymbol{D}\boldsymbol{V}\begin{bmatrix}x\\ y\\ z\end{bmatrix} \qquad (x,y,z)\in S_0. \tag{2-141}$$

Furthermore, since,

$$k^i\boldsymbol{V}^{-2} = k^i\,\mathrm{diag}\left(\frac{k_x^i}{k^i}, \frac{k_y^i}{k^i}, \frac{k_z^i}{k^i}\right) = \boldsymbol{k}^i, \tag{2-142}$$

substitution of (2-117) into (2-141) gives

$$f_P^- = 3\varepsilon\boldsymbol{J}^T((\boldsymbol{k}^m-\boldsymbol{k}^i)\boldsymbol{\Gamma}(\varrho e_0)-\boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m-\boldsymbol{k}^i))\boldsymbol{k}^i\boldsymbol{k}^m\boldsymbol{D}\begin{bmatrix}x\\ y\\ z\end{bmatrix}_{P\in S_0}. \tag{2-143}$$

Thus, $f_P^- = f_P^+$ and the condition in (2-10) is satisfied.

### 2.5.2 Special Cases of the Inclusion Problem

We will finally consider special cases of the generalized inclusion model in this study by varying the permeability contrast and the anisotropy ratio of the formation. Furthermore, we will show that particular inclusion problems reported in [1, 7] can be directly derived from (2-119) and

(2-120). Consider now the pressure distributions given by (2-119) and (2-120), and define the matrix

$$\boldsymbol{\delta} = (\boldsymbol{k}^m)^{-1}\boldsymbol{k}^i \qquad \text{i.e.,} \quad \boldsymbol{\delta} = \text{diag}(k_x^i/k_x^m, k_y^i/k_y^m, k_z^i/k_z^m), \tag{2-144}$$

such that

$$\Phi^i(x,y,z) = \boldsymbol{J}^T((\boldsymbol{I}-\boldsymbol{\delta})\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2-145}$$

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T((\boldsymbol{I}-\boldsymbol{\delta})\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1}(\boldsymbol{I}-\boldsymbol{\delta})\boldsymbol{\Gamma}(\acute{e})\begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2-146}$$

The matrix $\boldsymbol{\delta}$ expresses the degree of heterogeneity, we investigate therefore the behaviour of pressure distribution in $\Omega$ at certain extreme permeability contrasts.

- **Ellipsoidal inclusion surrounded by a highly permeable medium**: If the ellipsoidal inclusion is submerged in a medium of highly permeable rock (e.g. a sand body containing aligned ellipsoidal shale inclusions), then $\boldsymbol{\delta} \sim \boldsymbol{0}$. In this case, the distribution of pressure is,

$$\Phi^i(x,y,z) \approx \boldsymbol{J}^T(\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in } \Omega_0, \tag{2-147}$$

$$\Phi^m(x,y,z) \approx -\boldsymbol{J}^T\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T(\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1}\boldsymbol{\Gamma}(\acute{e})\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in } \Omega_1. \tag{2-148}$$

- **Ellipsoidal inclusion surrounded by an impermeable medium**: Consider now a case in which the inclusion is surrounded by a relatively impermeable matrix (e.g. a medium with fracture of high conductivity). In such a case, $\boldsymbol{\delta} \sim \infty$. Thus, by referring to (2-145) and (2-146), we have

$$((\boldsymbol{I}-\boldsymbol{\delta})\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1} \sim \boldsymbol{\delta}^{-1} \to \boldsymbol{0}, \tag{2-149}$$

$$((\boldsymbol{I}-\boldsymbol{\delta})\boldsymbol{\Gamma}(\acute{e}_0)-\boldsymbol{I})^{-1}(\boldsymbol{I}-\boldsymbol{\delta}) \sim (\boldsymbol{I}-\boldsymbol{\delta})^{-1}\boldsymbol{\Gamma}^{-1}(\acute{e}_0)(\boldsymbol{I}-\boldsymbol{\delta}) = \boldsymbol{\Gamma}^{-1}(\acute{e}_0). \tag{2-150}$$

Hence, the distribution of pressure reads

$$\Phi^i(x,y,z) \approx 0 \quad \text{in } \Omega_0, \quad \Phi^m(x,y,z) \approx -\boldsymbol{J}^T\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T\boldsymbol{\Gamma}^{-1}(\acute{e}_0)\boldsymbol{\Gamma}(\acute{e})\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{in } \Omega_1. \tag{2-151}$$

- **Homogeneous formation**: If we let the permeabilities of the ellipsoidal inclusion and the surrounding matrix be nearly equal in each of the principal directions, then $\boldsymbol{\delta} \sim \boldsymbol{I}$ and the medium should behave like a homogeneous formation. Hence,

$$\Phi^i(x,y,z) \sim \Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \qquad \text{in } \Omega. \tag{2-152}$$

- **Isotropic (spheroidal) inclusion submerged in an anisotropic matrix**: Consider now the distribution of pressure for a heterogeneous formation made up from the submersion of ellipsoidal inclusion of scalar permeability, in an anisotropic formation. This type of problem is reported solved by Dagan[1, 6]. Designating by $k^i$ the permeability of the spheroidal inclusion, we have $\boldsymbol{\delta} = k^i(\boldsymbol{k}^m)^{-1}$. Hence, it follows from (2-145) and (2-146) that

$$\Phi^i(x,y,z) = \boldsymbol{J}^T((\boldsymbol{k}^m - k^{-1}\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{k}^m)^{-1}\boldsymbol{k}^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2-153}$$

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T((\boldsymbol{k}^m - k^i\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{k}^m)^{-1}(\boldsymbol{k}^m - k^i\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2-154}$$

Assuming oblate spheroidal inclusion, see Dagan[1]

$$\boldsymbol{\Gamma}(\acute{e}) = v(\acute{e}) \begin{bmatrix} \frac{1}{2}\lambda(\acute{e}) & & \\ & \frac{1}{2}\lambda(\acute{e}) & \\ & & 1-\lambda(\acute{e}) \end{bmatrix}, \tag{2-155}$$

where $v(\acute{e})$ and $\lambda(\acute{e})$ are defined in (2-87, 2-89).

- **Isotropic ellipsoidal inclusion in an Isotropic formation**: We conclude this chapter by considering a simple case in which both the ellipsoidal inclusion and the surrounding matrix have scalar permeability. Denoting by $\sigma = k^i/k^m$ the permeability contrast, the distribution of internal and external pressures are easily derived from (2-145) and (2-146) by putting $\boldsymbol{\delta} = \sigma\boldsymbol{I}$. Thus,

$$\Phi^i(x,y,z) = \boldsymbol{J}^T((\boldsymbol{I} - \sigma\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I})^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2-156}$$

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{J}^T((\boldsymbol{I} - \sigma\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I})^{-1}(\boldsymbol{I} - \sigma\boldsymbol{I})\boldsymbol{\Gamma}(\acute{e}) \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2-157}$$

Similar results are reported in [7] in the context of determining the distribution of temperature in a heterogeneous material where $\sigma$ being thermal conductivity anisotropy ratio.

# Chapter 3

# Single Ellipsoidal Inclusion with Arbitrary Permeability Orientation

We have derived analytical solutions for the distribution of pressure by assuming that the permeability tensors of the inclusion and the surrounding matrix are aligned with the principal axes of the ellipsoid. In this chapter, we generalize the preceding solutions by allowing arbitrary orientation of the permeability tensors relative to the principal axes of the ellipsoid.

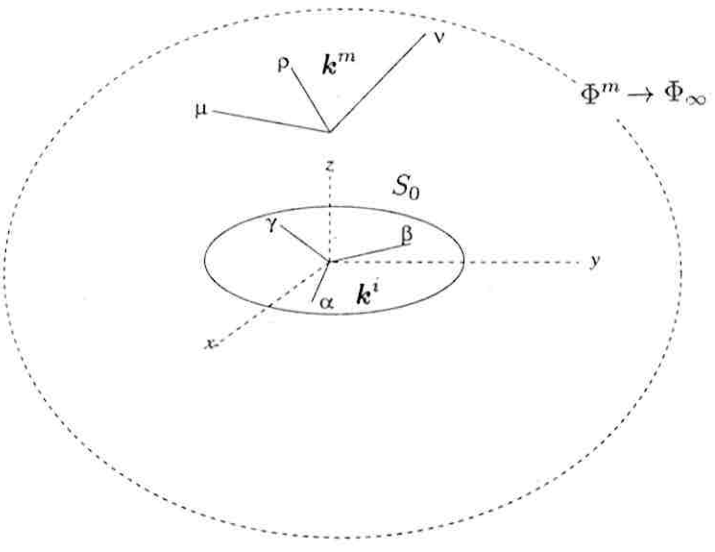



Figure 3-1: *An individual ellipsoidal inclusion with permeability tensors arbitrarily oriented relative to the principal axes of the ellipsoid.*

## 3.1 Formulation of the Problem

Consider the heterogeneous formation described in Section 2.1 and let now $(\alpha, \beta, \gamma)$ and $(\rho, \mu, \nu)$ be orthogonal coordinate systems in which the permeability tensors of the inclusion and the matrix are respectively diagonal. The internal and external potentials are now satisfied by

$$k^i_\alpha \frac{\partial^2 \Phi^i}{\partial \alpha^2} + k^i_\beta \frac{\partial^2 \Phi^i}{\partial \beta^2} + k^i_\gamma \frac{\partial^2 \Phi^i}{\partial \gamma^2} = 0 \qquad \text{(inside } S_0\text{)}\ , \tag{3-1}$$

$$k^m_\rho \frac{\partial^2 \Phi^m}{\partial \rho^2} + k^m_\mu \frac{\partial^2 \Phi^m}{\partial \mu^2} + k^m_\nu \frac{\partial^2 \Phi^m}{\partial \nu^2} = 0 \qquad \text{(outside } S_0\text{)}\ , \tag{3-2}$$

where $S_0$ denotes by the ellipsoidal surface in (2-1). Furthermore, designating now by $(J_\rho, J_\mu, J_\nu)$ the components of a constant potential gradient, we prescribe uniform flow at infinity i.e.,

$$\Phi^m(\rho,\mu,\nu) \rightarrow \Phi_\infty(\rho,\mu,\nu) = -\boldsymbol{J}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \qquad \text{as} \quad \|(\rho,\mu,\nu)\| \rightarrow \infty. \tag{3-3}$$

Finally, denoting by $P \in S_0$ an arbitrary surface point, we require continuity of potential and the normal flux on the surface of the inclusion,

$$\Phi^i_P = \Phi_P, \tag{3-4}$$
$$f^-_P = f^+_P, \tag{3-5}$$

where

$$f^- = k^i_\alpha \frac{\partial \Phi^i}{\partial \alpha} n_\alpha + k^i_\beta \frac{\partial \Phi^i}{\partial \beta} n_\beta + k^i_\gamma \frac{\partial \Phi^i}{\partial \gamma} n_\gamma, \tag{3-6}$$
$$f^+ = k^m_\rho \frac{\partial \Phi^m}{\partial \rho} n_\rho + k^m_\mu \frac{\partial \Phi^m}{\partial \mu} n_\mu + k^m_\nu \frac{\partial \Phi^m}{\partial \nu} n_\nu. \tag{3-7}$$

Here $(n_\alpha, n_\beta, n_\gamma)$ and $(n_\rho, n_\mu, n_\nu)$ are the components of the unit vector $\boldsymbol{n}$ normal to $S_0$.

## 3.2 Principal Axis Transformation

The most general equation of a second degree quadric in the variables, say $\rho$, $\mu$, $\nu$ may be written in the form

$$\begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}^T \boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} + 2\boldsymbol{a}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} + a = 0. \tag{3-8}$$

Here $\boldsymbol{A} \in R^{3\times3}$ is a real matrix whose off-diagonal elements (i.e., the terms which describe rotation relative to the coordinate axes) are not all zero, $\boldsymbol{a}$ is a real column vector which describes translation of the surface out of origo, and $a$ is an arbitrary constant. Assuming $\boldsymbol{A}$ be symmetric and positive definite i.e,

$$\boldsymbol{A} = \boldsymbol{A}^T; \qquad \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}^T \boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} > 0, \tag{3-9}$$

we may put

$$\boldsymbol{a} = 0, \tag{3-10}$$

such that, by taking the value of $a$ to be unity, the quadric in (3-8) describes an *inertia ellipsoid* in $(\rho, \mu, \nu)$, of equation

$$S_0: \qquad \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}^T \boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} = 1. \tag{3-11}$$

Similarly, denoting by $\boldsymbol{A}^i$ a real symmetric and positive definite matrix, the surface equation in $(\alpha, \beta, \gamma)$ is given by

$$S_0: \qquad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^T \boldsymbol{A}^i \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 1. \tag{3-12}$$

Our objective is to express the left-hand side of (3-11) and (3-12) as a sum of squares in $(x, y, z)$, such that the above surface equations take on the central quadric form in Chapter 2(2-1). For this purpose, denote by $(\boldsymbol{e}_\rho, \boldsymbol{e}_\mu, \boldsymbol{e}_\nu)$ and $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ unit vectors oriented along the axes of the coordinate systems $(\rho, \mu, \nu)$ and $(x, y, z)$, respectively, see Fig. [3-2].

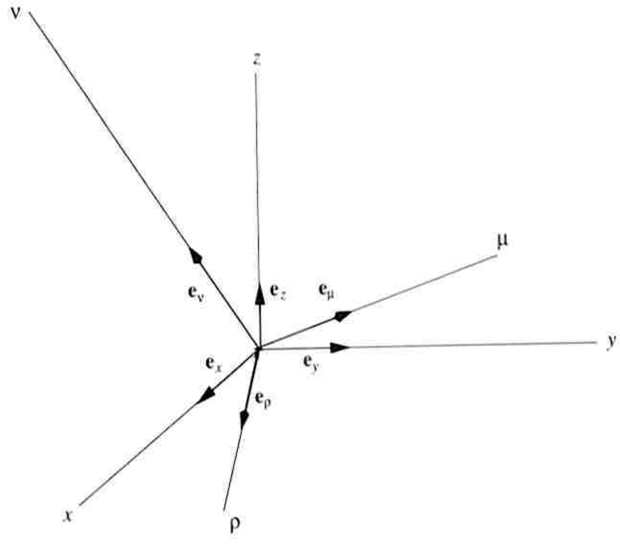



Figure 3-2: *Coordinate systems and unit vectors used to describe the orthogonal transformation.*

Furthermore, let the direction cosines specify the orientation of $(\rho, \mu, \nu)$ relative to $(x, y, z)$. Hence

$$\begin{aligned} \rho &= \cos(\boldsymbol{e}_\rho, \boldsymbol{e}_x)x + \cos(\boldsymbol{e}_\rho, \boldsymbol{e}_y)y + \cos(\boldsymbol{e}_\rho, \boldsymbol{e}_z)z \\ \mu &= \cos(\boldsymbol{e}_\mu, \boldsymbol{e}_x)x + \cos(\boldsymbol{e}_\mu, \boldsymbol{e}_y)y + \cos(\boldsymbol{e}_\mu, \boldsymbol{e}_z)z \\ \nu &= \cos(\boldsymbol{e}_\nu, \boldsymbol{e}_x)x + \cos(\boldsymbol{e}_\nu, \boldsymbol{e}_y)y + \cos(\boldsymbol{e}_\nu, \boldsymbol{e}_z)z \end{aligned} \tag{3-13}$$

is an orthogonal 3D axes rotation (origo unchanged) which preserves the the identity

$$||(\rho, \mu, \nu)|| = ||(x, y, z)||. \qquad \text{(inner product invariant)} \tag{3-14}$$

For convenience, let now $(l_r, m_r, n_r)$, $r = x, y, z$, denote by the nine direction cosines in (3-13), where

$$\begin{cases} l_r = \cos(\boldsymbol{e}_\rho, \boldsymbol{e}_r) \\ m_r = \cos(\boldsymbol{e}_\mu, \boldsymbol{e}_r) \\ n_r = \cos(\boldsymbol{e}_\nu, \boldsymbol{e}_r) \end{cases} \qquad r = x, y, z, \tag{3-15}$$

such that

$$\begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \boldsymbol{Q} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}. \tag{3-16}$$

Since the matrix $\boldsymbol{Q}$ must be orthogonal

$$\boldsymbol{Q}^T \boldsymbol{Q} = \boldsymbol{I}, \tag{3-17}$$

hence the reverse of the transformation in (3-16) is easily given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}. \tag{3-18}$$

Similarly, the transformation of the equation of the ellipsoid in $(\alpha, \beta, \gamma)$ into a central quadric in $(x, y, z)$ is accomplished by means of the orthogonal transformation

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \boldsymbol{Q}_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad , \qquad \boldsymbol{Q}_i = \begin{bmatrix} l^i_x & l^i_y & l^i_z \\ m^i_x & m^i_y & m^i_z \\ n^i_x & n^i_y & n^i_z \end{bmatrix}, \tag{3-19}$$

where $(l^i_r, m^i_r, n^i_r)$, $r = x, y, z$ are the direction cosines which specify the orientation of $(\alpha, \beta, \gamma)$ with respect to $(x, y, z)$ . Note that the rotation matrices $\boldsymbol{Q}$ and $\boldsymbol{Q}_i$ may be expressed as functions of either direction cosines or rotation angles (e.g., Euler's angles; see Appendix B). However, unless otherwise stated the dependence of the matrices on direction cosines or rotation angles will not be explicitly indicated.

Finally, by making use of the preceding orthogonal transformations, we establish a relationship between the coordinate systems $(\rho, \mu, \nu)$ and $(\alpha, \beta, \gamma)$ i.e, the orientation of permeability tensors of the inclusion and the surrounding matrix relative to each other. For instance, it follows from (3-16) and (3-19) that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{Q}_i^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}. \tag{3-20}$$

Thus

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \boldsymbol{Q}_i \boldsymbol{Q}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}. \tag{3-21}$$

Hence, by (3-11) that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^T \boldsymbol{Q}_i \boldsymbol{Q}^T \boldsymbol{A} \boldsymbol{Q} \boldsymbol{Q}_i^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 1. \tag{3-22}$$

By comparing above equation to (3-12) we obtain

$$\boldsymbol{A}^i = \boldsymbol{Q}_i \boldsymbol{Q}^T \boldsymbol{A} \boldsymbol{Q} \boldsymbol{Q}_i^T. \tag{3-23}$$

The relations in (3-21) and (3-23) will be used in the algebraic operations needed for the continuity conditions.

### 3.2.1 Orthogonal Transformation of Surface Equation

Let us now consider the transformation of the quadrics in (3-11) and (3-12). The key ingredient in the transformations is to orthogonally decompose the matrices $\boldsymbol{A}$ and $\boldsymbol{A}^i$. For instance, by substitution of (3-16) into (3-11),

$$S_0: \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{Q}^T \boldsymbol{A} \boldsymbol{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1. \tag{3-24}$$

Furthermore, let here $\lambda_r$, $(r = x, y, z)$ denote by the (positive and distinct) eigenvalues of $\boldsymbol{A}$. Designating by $\boldsymbol{q}_r \in \boldsymbol{Q}$, $(r = x, y, z)$, the basis of the three eigenvectors associated with $\lambda_x$, $\lambda_y$ and $\lambda_z$, where

$$\boldsymbol{q}_r = \begin{bmatrix} l_r \\ m_r \\ n_r \end{bmatrix} \qquad r = x, y, z, \tag{3-25}$$

then by a real Schur decomposition[18] of $\boldsymbol{A}$, there exists a diagonal matrix[18] whose entries are the matrix $\boldsymbol{Q}^T \boldsymbol{A} \boldsymbol{Q}$ is diagonalized i.e.,

$$\boldsymbol{q}_r^T \boldsymbol{A} \boldsymbol{q}_s = \begin{cases} 0 & \text{for } r \neq s, \\ \lambda_r & \text{for } r = s, \end{cases} \qquad r, s = x, y, z, \tag{3-26}$$

Denoting such matrix by $\boldsymbol{\lambda}$, the decomposition of $\boldsymbol{A}$ reads

$$\boldsymbol{A} = \boldsymbol{Q} \boldsymbol{\lambda} \boldsymbol{Q}^T; \qquad \boldsymbol{\lambda} = \mathrm{diag}(\lambda_x, \lambda_y, \lambda_z). \tag{3-27}$$

Accordingly, the central quadric form of (3-11) is

$$S_0: \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{\lambda} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_x x^2 + \lambda_y y^2 + \lambda_z z^2 = 1, \tag{3-28}$$

in which the principal axes of the ellipsoid are now aligned with the principal axes $x$, $y$ and $z$. For convenience, we may define

$$a = \frac{1}{\sqrt{\lambda_x}}; \qquad b = \frac{1}{\sqrt{\lambda_y}}; \qquad c = \frac{1}{\sqrt{\lambda_z}}, \tag{3-29}$$

where $a, b, c$ are the semi-axes of the ellipsoid, such that

$$S_0: \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{3-30}$$

Similarly, the orthogonal transformation in (3-19) puts the equation of the ellipsoid in $(\alpha, \beta, \gamma)$ into its central quadric form:

$$S_0: \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{\lambda}^i \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_x^i x^2 + \lambda_y^i y^2 + \lambda_z^i z^2 = 1, \tag{3-31}$$

where $\lambda_r^i$, $r = x, y, z$ are the eigenvalues of $\boldsymbol{A}^i$.

## 3.3 Isotropic Equivalents of Potential Problems

By direct application of (3-16) and (3-19), the orthogonal transformation of the differential equations in (3-2) and (3-1) may involve full permeability tensors. Therefore, we simplify the problem by first determining the isotropic equivalents of the potential problems by means of the following coordinates

$$\begin{bmatrix} \acute{\rho} \\ \acute{\mu} \\ \acute{\nu} \end{bmatrix} = \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}, \qquad \boldsymbol{W} = \mathrm{diag}(\sqrt{k^m/k_\rho^m}, \sqrt{k^m/k_\mu^m}, \sqrt{k^m/k_\nu^m}). \tag{3-32}$$

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{bmatrix} = \boldsymbol{V} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \qquad \boldsymbol{V} = \mathrm{diag}(\sqrt{k^i/k_\alpha^i}, \sqrt{k^i/k_\beta^i}, \sqrt{k^i/k_\gamma^i}). \tag{3-33}$$

Hence the external potential satisfies now

$$\frac{\partial^2 \Phi^m}{\partial \acute{\rho}^2}+\frac{\partial^2 \Phi^m}{\partial \acute{\mu}^2}+\frac{\partial^2 \Phi^m}{\partial \acute{\nu}^2}=0 \qquad \text{(outside } \acute{S}_0\text{);} \tag{3-34}$$

subject to the boundary condition

$$\Phi^m(\acute{\rho}, \acute{\mu}, \acute{\nu}) \to \Phi_\infty(\acute{\rho}, \acute{\mu}, \acute{\nu}) = -\acute{\boldsymbol{J}}^T \begin{bmatrix} \acute{\rho} \\ \acute{\mu} \\ \acute{\nu} \end{bmatrix} \qquad \text{as} \quad \|(\acute{\rho}, \acute{\mu}, \acute{\nu})\| \to \infty, \tag{3-35}$$

Similarly, the isotropic equivalent of the interior problem is

$$\frac{\partial^2 \Phi^i}{\partial \tilde{\alpha}^2}+\frac{\partial^2 \Phi^i}{\partial \tilde{\beta}^2}+\frac{\partial^2 \Phi^i}{\partial \tilde{\gamma}^2}=0 \qquad \text{(inside } \tilde{S}_0\text{).} \tag{3-36}$$

In the above equations, $\acute{S}_0$ and $\tilde{S}_0$ indicate the transforms of $S_0$ equations in (3-11) and (3-12),

$$\acute{S}_0: \qquad \begin{bmatrix} \acute{\rho} \\ \acute{\mu} \\ \acute{\nu} \end{bmatrix}^T \acute{\boldsymbol{A}} \begin{bmatrix} \acute{\rho} \\ \acute{\mu} \\ \acute{\nu} \end{bmatrix} = 1, \tag{3-37}$$

$$\tilde{S}_0: \qquad \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{bmatrix}^T \tilde{\boldsymbol{A}}^i \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{bmatrix} = 1. \tag{3-38}$$

$\acute{\boldsymbol{A}}$ and $\tilde{\boldsymbol{A}}^i$ are transforms of the coefficient matrices in (3-11) and (3-12).

## 3.4 Orthogonal Transformation of Isotropized Potential Problems

In virtue of the coordinate transformations in (3-32) and (3-33), the orthogonal transformations in (3-16) and (3-19) are now given by

$$\begin{bmatrix} \acute{\rho} \\ \acute{\mu} \\ \acute{\nu} \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{bmatrix} = \boldsymbol{Q}_i \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}. \tag{3-39}$$

Thus, the transformation of (3-34) reads

$$\frac{\partial^2 \Phi^m}{\partial \acute{x}^2} + \frac{\partial^2 \Phi^m}{\partial \acute{y}^2} + \frac{\partial^2 \Phi^m}{\partial \acute{z}^2} = 0 \qquad \text{(outside } \acute{S}_0\text{)}, \tag{3-40}$$

where

$$\acute{S}_0: \qquad \frac{\acute{x}^2}{\acute{a}^2} + \frac{\acute{y}^2}{\acute{b}^2} + \frac{\acute{z}^2}{\acute{c}^2} = 1, \tag{3-41}$$

and the boundary condition reads now

$$\Phi^m(\acute{x}, \acute{y}, \acute{z}) \to \Phi_\infty(\acute{x}, \acute{y}, \acute{z}) = -\acute{J}_x \acute{x} - \acute{J}_y \acute{y} - \acute{J}_z \acute{z} \qquad \text{as} \quad \|(\acute{x}, \acute{y}, \acute{z})\| \to \infty. \tag{3-42}$$

where

$$\begin{bmatrix} \acute{J}_x \\ \acute{J}_y \\ \acute{J}_z \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} \acute{J}_\rho \\ \acute{J}_\mu \\ \acute{J}_\nu \end{bmatrix}. \tag{3-43}$$

Similarly, the orthogonal transformation of the isotropized interior problem gives

$$\frac{\partial^2 \Phi^i}{\partial \tilde{x}^2} + \frac{\partial^2 \Phi^i}{\partial \tilde{y}^2} + \frac{\partial^2 \Phi^i}{\partial \tilde{z}^2} = 0 \qquad \text{(inside } \tilde{S}_0\text{)}, \tag{3-44}$$

where

$$\tilde{S}_0: \qquad \frac{\tilde{x}^2}{\tilde{c}^2} + \frac{\tilde{y}^2}{\tilde{b}^2} + \frac{\tilde{z}^2}{\tilde{c}^2} = 1. \tag{3-45}$$

## 3.5 Solution of Potential Problems

The potential problems in (3-40) and (3-44) are analogous[1] to the problems in the previous chapter, see (2-14, 2-20). Hence by straightforward reference to the solution methodology and

[1]The similarity in notation must not be confused.

the underlaying assumptions, possible solutions to (3-40) and (3-44) read, (refer to (2-43) and (2-17)) can be readily written down as

$$\Phi^i(\tilde{x},\tilde{y},\tilde{z}) = \boldsymbol{v}^T \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \qquad \text{(inside } \tilde{S}_0\text{)}, \qquad (3\text{-}46)$$

$$\Phi^m(\acute{x},\acute{y},\acute{z}) = -(\acute{J}_x,\acute{J}_y,\acute{J}_z) \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} + \boldsymbol{w}^T \boldsymbol{\Gamma}(\acute{e}) \begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} \qquad \text{(outside } \acute{S}_0\text{)}. \qquad (3\text{-}47)$$

Switching to the coordinates $(\alpha,\beta,\gamma)$ and $(\rho,\mu,\nu)$, the potentials read

$$\Phi^i(\alpha,\beta,\gamma) = \boldsymbol{v}^T \boldsymbol{Q}_i^T \boldsymbol{V} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad \text{in } \Omega_0, \qquad (3\text{-}48)$$

$$\Phi^m(\rho,\mu,\nu) = -\boldsymbol{J}^T \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} + \boldsymbol{w}^T \boldsymbol{\Gamma}(\acute{e}) \boldsymbol{Q}^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \qquad \text{in } \Omega_1, \qquad (3\text{-}49)$$

where $\boldsymbol{V}$ and $\boldsymbol{W}$ are now defined in (3-32) and (3-33), respectively. $\boldsymbol{v}$ and $\boldsymbol{v}$ are constant vectors to be determined by satisfying the continuity conditions in (3-4) and (3-5), and $\boldsymbol{\Gamma}(\acute{e})$ is a diagonal matrix whose entries are the associated exterior harmonics given by the following elliptic integrals

$$\begin{aligned}
\Gamma^x(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{\sqrt{(1-u^2)[1-\acute{\kappa}(1-u^2)]}}\, du, \\
\Gamma^y(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{[1-\acute{\kappa}^2(1-u^2)]\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\, du. \qquad (3\text{-}50)\\
\Gamma^z(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{u^2\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\, du.
\end{aligned}$$

Here

$$\acute{\Theta}_\kappa = \frac{\delta e_0}{(1-\delta^2 e_0^2)^{\frac{3}{2}}} \sqrt{1-\acute{\kappa}^2(1-\delta^2 e_0^2)}, \qquad \acute{\kappa}^2 = \frac{k_\mu^m - \delta^2 E_0^2}{k_\mu^m - \delta^2 e_0^2}. \qquad (3\text{-}51)$$

and

$$\delta = \sqrt{k_\rho^m / k_\nu^m}. \qquad (3\text{-}52)$$

Alternatively, the pressure distributions may be expressed in terms of the $(x,y,z)$ coordinates system. Hence,

$$\Phi^i(x,y,z) = \boldsymbol{v}^T \boldsymbol{Q}_i^T \boldsymbol{V} \boldsymbol{Q}_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in}\Omega_0, \qquad (3\text{-}53)$$

$$\Phi^m(x,y,z) = -\boldsymbol{J}^T \boldsymbol{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{w}^T \boldsymbol{\Gamma}(\acute{e}) \boldsymbol{Q}^T \boldsymbol{W} \boldsymbol{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in } \Omega_1, \qquad (3\text{-}54)$$

However, it is emphasized that the above elliptic integrals implicitly capture the transformations in (3-32) and the first equation in (3-39) by means of eccentricities. Therefore, to evaluate the associated external harmonics, we derive transformations formulae for eccentricities.

## 3.6 Transformation of Eccentricities

Since $e_0$ and $E_0$ denote by the meridian and equatorial eccentricities of the ellipsoid, we may first derive expressions for $\acute{e}_0$ and $\acute{E}_0$. For instance, by elimination of the vector $(\acute{\rho}, \acute{\mu}, \acute{\nu})^T$ from (3-32) and (3-39), we obtain

$$\boldsymbol{Q}\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = \boldsymbol{W}\begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}. \tag{3-55}$$

Furthermore, by substitution of (3-16) into the above equation, we multiply both sides of the equation by $\boldsymbol{Q}^T$ and obtain the coordinate transformation

$$\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = \boldsymbol{Q}^T\boldsymbol{W}\boldsymbol{Q}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{3-56}$$

from which

$$\begin{bmatrix} \acute{a} \\ \acute{b} \\ \acute{c} \end{bmatrix} = \boldsymbol{Q}^T\boldsymbol{W}\boldsymbol{Q}\begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{3-57}$$

Hence the ratios

$$\frac{\acute{c}}{\acute{a}} = \frac{\boldsymbol{q}_z^T\boldsymbol{W}\boldsymbol{q}_x a + \boldsymbol{q}_z^T\boldsymbol{W}\boldsymbol{q}_y b + \boldsymbol{q}_z^T\boldsymbol{W}\boldsymbol{q}_z c}{\boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_x a + \boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_y b + \boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_z c}, \tag{3-58}$$

$$\frac{\acute{b}}{\acute{a}} = \frac{\boldsymbol{q}_y^T\boldsymbol{W}\boldsymbol{q}_x a + \boldsymbol{q}_y^T\boldsymbol{W}\boldsymbol{q}_y b + \boldsymbol{q}_y^T\boldsymbol{W}\boldsymbol{q}_z c}{\boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_x a + \boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_y b + \boldsymbol{q}_x^T\boldsymbol{W}\boldsymbol{q}_z c}, \tag{3-59}$$

define the required transforms of $e_0, E_0$. Thus, by the definition of meridian and equatorial eccentricities

$$\acute{e}_0 = \frac{\boldsymbol{q}_z^T\boldsymbol{\omega}\boldsymbol{q}_x + \boldsymbol{q}_z^T\boldsymbol{\omega}\boldsymbol{q}_y E_0 + \boldsymbol{q}_z^T\boldsymbol{\omega}\boldsymbol{q}_z e_0}{\boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_x + \boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_y E_0 + \boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_z e_0}, \tag{3-60}$$

$$\acute{E}_0 = \frac{\boldsymbol{q}_y^T\boldsymbol{\omega}\boldsymbol{q}_x + \boldsymbol{q}_y^T\boldsymbol{\omega}\boldsymbol{q}_y E_0 + \boldsymbol{q}_y^T\boldsymbol{\omega}\boldsymbol{q}_z e_0}{\boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_x + \boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_y E_0 + \boldsymbol{q}_x^T\boldsymbol{\omega}\boldsymbol{q}_z e_0}, \tag{3-61}$$

where we defined

$$\omega = \mathrm{diag}(1, \sqrt{\frac{k_\rho^m}{k_\mu^m}}, \sqrt{\frac{k_\rho^m}{k_\nu^m}}), \qquad \boldsymbol{\omega} = \sqrt{\frac{k_\rho^m}{k^m}}\boldsymbol{W}. \tag{3-62}$$

It may be seen that when the components of the permeability tensor $\boldsymbol{k}^m$ are aligned with the principal axes of the ellipsoid, $\boldsymbol{Q} = \boldsymbol{I}$ i.e.,

$$q_x^T = (1,0,0), \qquad q_y^T = (0,1,0), \qquad q_z^T = (0,0,1). \tag{3-63}$$

Thus, as expected, the transforms of eccentricities are

$$\acute{e}_0 = e_0\sqrt{\frac{k_\rho^m}{k_\nu^m}}, \qquad \acute{E}_0 = E_0\sqrt{\frac{k_\rho^m}{k_\mu^m}}. \tag{3-64}$$

## 3.7 Continuity Conditions

Let us finally consider to determine $\boldsymbol{v}$ and $\boldsymbol{w}$ in (3-48) and (3-49) by requiring th continuity of potential and the normal flux, see (3-4 and 3-5). Obviously, those conditions must be satisfied in any coordinate system. It is however emphasized that the functions in $\boldsymbol{\Gamma}(\acute{e})$ are defined in the $(\acute{x}, \acute{y}, \acute{z})$ coordinate system. Hence we will switch to that coordinate system to easily carry out the differentiations of $\boldsymbol{\Gamma}(\acute{e})$ required for the derivations related to the normal derivative.

### 3.7.1 Continuity of Potential

By the condition of the continuity of potential (see 3-4),

$$(\boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{e}_0)\boldsymbol{Q}^T\boldsymbol{W} - \boldsymbol{J}^T)\begin{bmatrix}\rho\\ \mu\\ \nu\end{bmatrix}_{P\in S_0} = \boldsymbol{v}^T\boldsymbol{Q}_i^T\boldsymbol{V}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}_{P\in S_0}. \tag{3-65}$$

Furthermore, by the transformation (3-21), the above equation may be written as

$$(\boldsymbol{v}^T\boldsymbol{Q}_i^T\boldsymbol{V}\boldsymbol{Q}_i\boldsymbol{Q}^T - \boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{e}_0)\boldsymbol{Q}^T\boldsymbol{W} + \boldsymbol{J}^T)\begin{bmatrix}\rho\\ \mu\\ \nu\end{bmatrix}_{P\in S_0} = 0. \tag{3-66}$$

To satisfy this equation at $P \in S_0$, we must have (refer Section 2.4)

$$\boldsymbol{v}^T\boldsymbol{Q}_i^T\boldsymbol{V}\boldsymbol{Q}_i\boldsymbol{Q}^T - \boldsymbol{w}^T\boldsymbol{\Gamma}(\acute{e}_0)\boldsymbol{Q}^T\boldsymbol{W} + \boldsymbol{J}^T = \boldsymbol{0} \tag{3-67}$$

### 3.7.2 Continuity of the Normal Flux

By the surface equation in (3-12), the unit vector normal to $S_0$ may be written as

$$\boldsymbol{n}(\alpha,\beta,\gamma) = \varepsilon\boldsymbol{A}^i\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \qquad (\alpha,\beta,\gamma)\in S_0. \tag{3-68}$$

Hence the condition in (3-5) reads

$$f_P^- = \varepsilon\boldsymbol{v}^T\boldsymbol{Q}_i^T\boldsymbol{k}^i\boldsymbol{V}\boldsymbol{A}^i\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}_{P(\rho,\mu,\nu)\in S_0} \tag{3-69}$$

Furthermore, we use the relations in (3-21) and (3-23) and find the following expression for $f_P^-$

$$f_P^- = \varepsilon \boldsymbol{v}^T \boldsymbol{Q}_i^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{Q}_i \boldsymbol{Q}^T \boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix}_{P(\rho,\mu,\nu) \in S_0} \tag{3-70}$$

To determine $f_P^+$, we may first write (3-49) in the form

$$\Phi^m = \Phi_\infty + A\boldsymbol{q}_x^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^x(\acute{e}) + B\boldsymbol{q}_y^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^y(\acute{e}) + C\boldsymbol{q}_z^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^z(\acute{e}), \tag{3-71}$$

Differentiation of the above equation, first with respect to $\rho$, gives

$$\begin{aligned} \frac{\partial \Phi^m}{\partial \rho} &= -J_\rho + A\frac{\partial}{\partial \rho}(\boldsymbol{q}_x^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^x) + B\frac{\partial}{\partial \rho}(\boldsymbol{q}_y^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^y) + C\frac{\partial}{\partial \rho}(\boldsymbol{q}_z^T \boldsymbol{W} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \Gamma^z), \\ &= -J_\rho + Al_x\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \rho}(\rho\Gamma^x) + Bl_y\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \rho}(\rho\Gamma^y) + Cl_z\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \rho}(\rho\Gamma^z), \\ &= -J_\rho + Al_x\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^x) + Bl_y\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^y) + Cl_z\sqrt{\frac{k^m}{k_\rho^m}}\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^z), \end{aligned} \tag{3-72}$$

where

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^x) = \Gamma^x + (l_x\acute{x} + l_y\acute{y} + l_z\acute{z})(\frac{\partial \Gamma^x}{\partial \acute{x}}\frac{\partial \acute{x}}{\partial \acute{\rho}} + \frac{\partial \Gamma^y}{\partial \acute{y}}\frac{\partial \acute{x}}{\partial \acute{\rho}} + \frac{\partial \Gamma^z}{\partial \acute{z}}\frac{\partial \acute{x}}{\partial \acute{\rho}}), \tag{3-73}$$

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\mu}\Gamma^x) = \Gamma^x + (m_x\acute{x} + m_y\acute{y} + m_z\acute{z})(\frac{\partial \Gamma^x}{\partial \acute{x}}\frac{\partial \acute{x}}{\partial \acute{\mu}} + \frac{\partial \Gamma^y}{\partial \acute{y}}\frac{\partial \acute{x}}{\partial \acute{\mu}} + \frac{\partial \Gamma^z}{\partial \acute{z}}\frac{\partial \acute{x}}{\partial \acute{\mu}}), \tag{3-74}$$

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\nu}\Gamma^x) = \Gamma^x + (n_x\acute{x} + n_y\acute{y} + n_z\acute{z})(\frac{\partial \Gamma^x}{\partial \acute{x}}\frac{\partial \acute{x}}{\partial \acute{\nu}} + \frac{\partial \Gamma^y}{\partial \acute{y}}\frac{\partial \acute{x}}{\partial \acute{\nu}} + \frac{\partial \Gamma^z}{\partial \acute{z}}\frac{\partial \acute{x}}{\partial \acute{\nu}}). \tag{3-75}$$

Since we assume linearity of potential in $\boldsymbol{J}$, we may consider separate distributions of pressure. For instance, for flow in $(J_\rho, 0, 0)$ we may evaluate the above equations for the gradients in $(\acute{J}_x, 0, 0)$, $(0, \acute{J}_y, 0)$ and $(0, 0, \acute{J}_z)$. Thus, by first considering distribution of potential in $(\acute{J}_x, 0, 0)$, we obtain

$$\acute{\rho} = l_x\acute{x}, \qquad \acute{\mu} = m_x\acute{x}, \qquad \acute{\nu} = n_x\acute{x}. \tag{3-76}$$

Hence, by (3-73),

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^x) = \Gamma^x + \acute{x}\frac{\partial \Gamma^x}{\partial \acute{x}}, \tag{3-77}$$

Evaluation of this equation on the surface of the inclusion yields, as noted in Chapter 2,

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^x) = \Gamma^x(\acute{e}_0) - 1, \qquad \text{in } (0, \acute{J}_x, 0, 0). \tag{3-78}$$

Determining the remaining terms by symmetry

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^y) = \Gamma^y(\acute{e}_0) - 1 \qquad \text{in } (0, \acute{J}_y, 0), \tag{3-79}$$

$$\frac{\partial}{\partial \acute{\rho}}(\acute{\rho}\Gamma^z) = \Gamma^z(\acute{e}_0) - 1 \qquad \text{in } (0, 0, \acute{J}_z), \tag{3-80}$$

we substitute (3-78)-(3-80) into (3-72) and get

$$k_\rho^m \frac{\partial \Phi^m}{\partial \rho} = -J_\rho k_\rho^m + Al_x k_\rho^m \sqrt{\frac{k^m}{k_\rho^m}}(\Gamma^x(\acute{e}_0) - 1) + Bl_y k_\rho^m \sqrt{\frac{k^m}{k_\rho^m}}(\Gamma^y(\acute{e}_0) - 1) + Cl_z k_\rho^m \sqrt{\frac{k^m}{k_\rho^m}}(\Gamma^z(\acute{e}_0) - 1). \tag{3-81}$$

Accordingly, the remaining components of the normal derivative read

$$k_\mu^m \frac{\partial \Phi^m}{\partial \mu} = -J_\mu k_\mu^m + Am_x k_\mu^m \sqrt{\frac{k^m}{k_\mu^m}}(\Gamma^x(\acute{e}_0) - 1) + Bm_y k_\mu^m \sqrt{\frac{k^m}{k_\mu^m}}(\Gamma^y(\acute{e}_0) - 1) + Cm_z k_\mu^m \sqrt{\frac{k^m}{k_\mu^m}}(\Gamma^z(\acute{e}_0) - 1), \tag{3-82}$$

$$k_\nu^m \frac{\partial \Phi^m}{\partial \nu} = -J_\nu k_\nu^m + An_x k_\nu^m \sqrt{\frac{k^m}{k_\nu^m}}(\Gamma^x(\acute{e}_0) - 1) + Bn_y k_\nu^m \sqrt{\frac{k^m}{k_\nu^m}}(\Gamma^y(\acute{e}_0) - 1) + Cn_z k_\nu^m \sqrt{\frac{k^m}{k_\nu^m}}(\Gamma^z(\acute{e}_0) - 1). \tag{3-83}$$

Furthermore, since now

$$\boldsymbol{n}(\rho, \mu, \nu) = \varepsilon \boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \qquad (\rho, \mu, \nu) \in S_0, \tag{3-84}$$

we obtain the required expression for $f_P^+$

$$f_P^+ = \varepsilon(\boldsymbol{w}^T \boldsymbol{Q}^T \boldsymbol{k}^m \boldsymbol{W}(\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I}) - \boldsymbol{J}^T \boldsymbol{k}^m)\boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} \qquad (\rho, \mu, \nu) \in S_0, \tag{3-85}$$

and the condition $f_P^- = f_P^+$ leads to

$$(\boldsymbol{v}^T \boldsymbol{Q}_i^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{Q}_i \boldsymbol{Q}^T - \boldsymbol{w}^T \boldsymbol{Q}^T \boldsymbol{k}^m \boldsymbol{W}(\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I}) + \boldsymbol{J}^T \boldsymbol{k}^m)\boldsymbol{A} \begin{bmatrix} \rho \\ \mu \\ \nu \end{bmatrix} = 0. \qquad P(\rho, \mu, \nu) \in S_0 \tag{3-86}$$

To satisfy the above equation, we must have

$$\boldsymbol{v}^T \boldsymbol{Q}_i^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{Q}_i \boldsymbol{Q}^T - \boldsymbol{w}^T \boldsymbol{Q}^T \boldsymbol{k}^m \boldsymbol{W}(\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I}) + \boldsymbol{J}^T \boldsymbol{k}^m = \boldsymbol{0} \tag{3-87}$$

Finally, we transpose the equations in (3-67) and (3-87), such that $\boldsymbol{v}$ and $\boldsymbol{w}$ are determined by solving

$$\begin{bmatrix} \boldsymbol{Q}\boldsymbol{Q}_i^T \boldsymbol{V} \boldsymbol{Q}_i & -\boldsymbol{W}\boldsymbol{Q}\boldsymbol{\Gamma}(\acute{e}_0) \\ \boldsymbol{Q}\boldsymbol{Q}_i^T \boldsymbol{k}^i \boldsymbol{V} \boldsymbol{Q}_i & -\boldsymbol{k}^m \boldsymbol{W}\boldsymbol{Q}(\boldsymbol{\Gamma}(\acute{e}_0) - \boldsymbol{I}) \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{J} \\ -\boldsymbol{k}^m \boldsymbol{J} \end{bmatrix}. \tag{3-88}$$

The analytical solutions derived in this chapter complete the proposed generalization of the potential problems for single ellipsoidal inclusion embedded in homogeneous matrix of infinite dimensions.

# Chapter 4

# Composite Ellipsoidal Inclusions

Finally, we consider the pressure fluctuation created by a composite ellipsoidal inclusion, submerged in an infinite homogeneous matrix. By referring to the preceding chapters, recall that the exterior space was conveniently handled by ellipsoidal coordinates. Probably, we might be able to model the potential of an intermediate region, geometrically sandwiched between the interior ellipsoid and the exterior ellipsoid of a confocal system of composite ellipsoids.

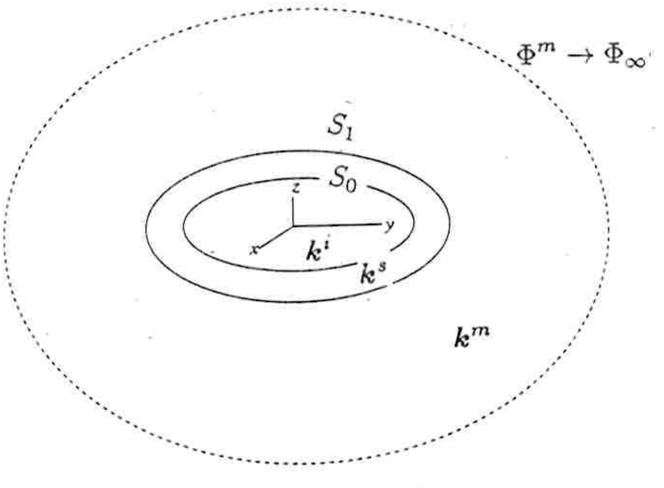



Figure 4-1: *Composite ellipsoidal inclusion model with permeability tensors aligned with principal axes of the confocal ellipsoids.*

The analysis proceeds along the same lines as in the preceding chapter. As shown in Fig. [4-1], the composite body is composed of an inner ellipsoid of tensorial permeability $\boldsymbol{k}^i$, surrounded by an ellipsoidal skin of tensorial permeability $\boldsymbol{k}^s$, sandwiched between the surfaces $S_0$ and $S_1$. The composite ellipsoid is then submerged in an infinite homogeneous matrix of tensorial permeability $\boldsymbol{k}^m$. For simplicity, we restrict our attention to a case in which all permeability tensors are aligned with the principal axes of the composite inclusion.

## 4.1 Formulation of the Problem

Still denoting by $\Omega \subset R^3$ the flow domain of infinite dimensions, let $\Omega_0$ be the region interior to $S_0$, $\Omega_1$ the ellipsoidal ring externally bounded by $S_1$, and $\Omega_2$ the region outside the surface $S_1$. Our purpose is to determine the internal, skin and external potential denoted by $\Phi^i$, $\Phi^s$ and $\Phi^m$, respectively, by solving the following set of partial differential equations satisfied by the potentials

$$k_x^i \frac{\partial^2 \Phi^i}{\partial x^2} + k_y^i \frac{\partial^2 \Phi^i}{\partial y^2} + k_z^i \frac{\partial^2 \Phi^i}{\partial z^2} = 0 \quad \text{in } \Omega_0, \tag{4-1}$$

$$k_x^s \frac{\partial^2 \Phi^s}{\partial x^2} + k_y^s \frac{\partial^2 \Phi^s}{\partial y^2} + k_z^s \frac{\partial^2 \Phi^s}{\partial z^2} = 0 \quad \text{in } \Omega_1, \tag{4-2}$$

$$k_x^m \frac{\partial^2 \Phi^m}{\partial x^2} + k_y^m \frac{\partial^2 \Phi^m}{\partial y^2} + k_z^m \frac{\partial^2 \Phi^m}{\partial z^2} = 0 \quad \text{in } \Omega_2, \tag{4-3}$$

Here the superscripts $s$ signify the skin problem. Assuming uniform flow at infinity, we prescribe the boundary condition

$$\Phi^m(x,y,z) \to \Phi_\infty(x,y,z) = -\boldsymbol{J}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{as} \quad \|(x,y,z)\| \to \infty. \tag{4-4}$$

Finally, we require the continuity of potential and the normal flux across the surfaces $S_0$ and $S_1$

$$\Phi^i_{P_0} = \Phi^s_{P_0}, \tag{4-5}$$

$$\left(\boldsymbol{k}^i \frac{\partial \Phi^i}{\partial n}\right)_{P_0} = \left(\boldsymbol{k}^s \frac{\partial \Phi^s}{\partial n}\right)_{P_0}, \tag{4-6}$$

$$\Phi^s_{P_1} = \Phi_{P_1}, \tag{4-7}$$

$$\left(\boldsymbol{k}^s \frac{\partial \Phi^s}{\partial m}\right)_{P_1} = \left(\boldsymbol{k}^m \frac{\partial \Phi^m}{\partial m}\right)_{P_1}. \tag{4-8}$$

Here $P_0 \in S_0$ and $P_1 \in S_1$ are arbitrary surface points, and $n$ and $m$ denote by the outward normals to $P_0 \in S_0$ and $P_1 \in S_1$, where

$$S_0: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D}_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \qquad \boldsymbol{D}_0 = \text{diag}(1/a_0^2, 1/b_0^2, 1/c_0^2), \tag{4-9}$$

$$S_1: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D}_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \qquad \boldsymbol{D}_1 = \text{diag}(1/a_1^2, 1/b_1^2, 1/c_1^2). \tag{4-10}$$

## 4.2 The Interior and Exterior Problems

The internal and external problems in (4-1) and (4-3) are similar to those solved previously. Hence, by referring to the solution methodology and the underlaying assumptions, we may readily write down the internal and external potentials

$$\Phi^i(x,y,z) = \boldsymbol{u}^T \boldsymbol{U} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in } \Omega_0, \tag{4-11}$$

$$\Phi^m(x,y,z) = \Phi_\infty(x,y,z) + \boldsymbol{w}^T \boldsymbol{\Gamma}(\acute{e}|\acute{\kappa}) \boldsymbol{W} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{in } \Omega_2, \tag{4-12}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are arbitrary constant vectors to be determined by continuity conditions. $\boldsymbol{U}$ and $\boldsymbol{W}$ are the matrices

$$\begin{aligned} \boldsymbol{U} &= \operatorname{diag}\left(\sqrt{\frac{k^i}{k^i_x}}, \sqrt{\frac{k^i}{k^i_y}}, \sqrt{\frac{k^i}{k^i_z}}\right), & (4\text{-}13) \\ \boldsymbol{W} &= \operatorname{diag}\left(\sqrt{\frac{k^m}{k^m_x}}, \sqrt{\frac{k^m}{k^m_y}}, \sqrt{\frac{k^m}{k^m_z}}\right), & (4\text{-}14) \end{aligned}$$

and finally, $\boldsymbol{\Gamma}(\acute{e})$ is the diagonal matrix whose entries are

$$\begin{aligned} \Gamma^x(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\, du, \\ \Gamma^y(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{[1-\acute{\kappa}^2(1-u^2)]\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\, du, \\ \Gamma^z(\acute{e}|\acute{\kappa}) &= \acute{\Theta}_\kappa \int_{\acute{e}}^1 \frac{1-u^2}{u^2\sqrt{(1-u^2)[1-\acute{\kappa}^2(1-u^2)]}}\, du. \end{aligned} \tag{4-15}$$

These integrals, in which

$$\acute{\Theta}_\kappa = \frac{\varrho e_1}{(1-\varrho^2 e_1^2)^{\frac{3}{2}}} \sqrt{1-\acute{\kappa}^2(1-\varrho^2 e_1^2)}, \tag{4-16}$$

$$\acute{\kappa}^2 = \frac{k_y - \varrho^2 E_1^2}{k_y - \varrho^2 e_1^2}, \qquad \varrho = \sqrt{k^m_x / k^m_z}, \tag{4-17}$$

are now to be evaluated for $\acute{e}_1 \leq \acute{e} \leq 1$.

## 4.3 The Skin Problem

By the coordinate transformation

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \boldsymbol{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \boldsymbol{V} = \operatorname{diag}\left(\sqrt{\frac{k^s}{k^s_x}}, \sqrt{\frac{k^s}{k^s_y}}, \sqrt{\frac{k^s}{k^s_z}}\right), \tag{4-18}$$

the isotropic equivalent of (4-2) reads

$$\frac{\partial^2 \Phi^s}{\partial \tilde{x}^2} + \frac{\partial^2 \Phi^s}{\partial \tilde{y}^2} + \frac{\partial^2 \Phi^s}{\partial \tilde{z}^2} = 0 \qquad \text{in } \tilde{\Omega}_1. \tag{4-19}$$

where $\tilde{\Omega}_1$ is the region bounded by the surfaces

$$\tilde{S}_0 : \qquad \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}^T \tilde{\boldsymbol{D}}_0 \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = 1, \qquad \tilde{\boldsymbol{D}}_0 = \text{diag}(1/\tilde{a}_0^2, 1/\tilde{b}_0^2, 1/\tilde{c}_0^2). \tag{4-20}$$

$$\tilde{S}_1 : \qquad \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}^T \tilde{\boldsymbol{D}}_1 \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = 1, \qquad \tilde{\boldsymbol{D}}_1 = \text{diag}(1/\tilde{a}_1^2, 1/\tilde{b}_1^2, 1/\tilde{c}_1^2). \tag{4-21}$$

We now refer to our discussion of ellipsoidal harmonics. Since the ellipsoidal harmonics for an intermediate ellipsoids could be constructed by linear combination of Lamé's first and second solution (see Section 1.2), a possible solution to the skin problem in (4-19) may have the form

$$\Phi^s = c_1 \Phi_1^s + c_2 \Phi_1^s \int_{\tilde{\xi}}^{\tilde{\xi}_1} \frac{du}{\chi^2(u)\sqrt{(\tilde{a}^2+u)(\tilde{b}^2+u)(\tilde{c}^2+u)}}, \tag{4-22}$$

where $\tilde{\xi}_1 =$ const. denotes by the ellipsoid of surface $\tilde{S}_1$. Thus, since we assume the ellipsoidal harmonics of second species form the basis of a possible solution

$$\Phi^s = \boldsymbol{v}_1^T \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} + \boldsymbol{v}_2^T \boldsymbol{\Gamma}(\tilde{e}|\tilde{\kappa}) \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \qquad \text{in } \tilde{\Omega}_1. \tag{4-23}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the arbitrary matrix vectors to be determined by the continuity conditions, and $\boldsymbol{\Gamma}(\tilde{e}|\tilde{\kappa})$ is the diagonal matrix whose entries are

$$\begin{aligned}
\Gamma^x(\tilde{e}|\tilde{\kappa}) &= \tilde{\Theta}_\kappa \int_{\tilde{e}}^{\tilde{e}_1} \frac{1-u^2}{\sqrt{(1-u^2)[1-\tilde{\kappa}^2(1-u^2)]}}\,du, \\
\Gamma^y(\tilde{e}|\tilde{\kappa}) &= \tilde{\Theta}_\kappa \int_{\tilde{e}}^{\tilde{e}_1} \frac{1-u^2}{[1-\tilde{\kappa}^2(1-u^2)]\sqrt{(1-u^2)[1-\tilde{\kappa}^2(1-u^2)]}}\,du, \\
\Gamma^z(\tilde{e}|\tilde{\kappa}) &= \tilde{\Theta}_\kappa \int_{\tilde{e}}^{\tilde{e}_1} \frac{1-u^2}{u^2\sqrt{(1-u^2)[1-\tilde{\kappa}^2(1-u^2)]}}\,du,
\end{aligned} \tag{4-24}$$

where, by noting the relations,

$$\tilde{e}_0 = e_0 \varrho_s; \qquad \tilde{e}_1 = e_1 \varrho_s, \tag{4-25}$$

$\Theta_{\tilde{\kappa}}$ and $\tilde{\kappa}$ are given by

$$\tilde{\Theta}_\kappa = \frac{\varrho_s e_0}{(1-\varrho_s^2 e_0^2)^{\frac{3}{2}}} \sqrt{1-\tilde{\kappa}^2(1-\varrho_s^2 e_0^2)}, \tag{4-26}$$

$$\tilde{\kappa}^2 = \frac{k_y^s - \varrho_s^2 E_0^2}{k_y^s - \varrho_s^2 e_0^2}, \qquad \varrho_s = \sqrt{k_x^s / k_z^s}. \tag{4-27}$$

Switching to the $(x, y, z)$ coordinates, we obtain

$$\Phi^s(x,y,z) = \boldsymbol{v}_1^T \boldsymbol{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{v}_2^T \boldsymbol{\Gamma}(\tilde{e}|\tilde{\kappa}) \boldsymbol{V} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (\tilde{e}_0 \leq \tilde{e} \leq \tilde{e}_0); \qquad \text{in } \Omega_1. \tag{4-28}$$

## 4.4 Volume Ratios Between Composite Ellipsoids

Consider the ellipsoidal ring between $\tilde{S}_0$ and $\tilde{S}_1$. The equation

$$\tilde{S}: \quad \frac{\tilde{x}^2}{\tilde{a}_0^2 + \tilde{\xi}} + \frac{\tilde{y}^2}{\tilde{b}_0^2 + \tilde{\xi}} + \frac{\tilde{z}^2}{\tilde{c}_0^2 + \tilde{\xi}} = 1; \qquad 0 \leq \tilde{\xi} \leq \tilde{\xi}_1, \tag{4-29}$$

describes the surfaces of the ellipsoids $\tilde{\xi}$ =const. in $\tilde{\Omega}_1$, and

$$\tilde{e} = \left(\frac{\tilde{c}_0^2 + \tilde{\xi}}{\tilde{a}_0^2 + \tilde{\xi}}\right)^{\frac{1}{2}}; \qquad \tilde{E} = \left(\frac{\tilde{b}_0^2 + \tilde{\xi}}{\tilde{a}_0^2 + \tilde{\xi}}\right)^{\frac{1}{2}}, \tag{4-30}$$

accordingly are the transforms of eccentricities. Since $\tilde{\xi} = 0$ and $\tilde{\xi} = \tilde{\xi}_1$ define the ellipsoids in (4-20) and (4-21), respectively, the volumes $\tilde{V}_0$ and $\tilde{V}_1$ for the interior and exterior ellipsoid are

$$\tilde{V}_0 = \frac{4\pi}{3}\tilde{a}_0\tilde{b}_0\tilde{c}_0, \qquad \tilde{V}_1 = \frac{4\pi}{3}\sqrt{(\tilde{a}_0^2 + \tilde{\xi}_1)(\tilde{b}_0^2 + \tilde{\xi}_1)(\tilde{c}_0^2 + \tilde{\xi}_1)}. \tag{4-31}$$

Hence the volume ratio of the interior ellipsoid; denoted by $\alpha$ is

$$\alpha = \frac{\tilde{a}_0\tilde{b}_0\tilde{c}_0}{\sqrt{(\tilde{a}_0^2 + \tilde{\xi}_1)(\tilde{b}_0^2 + \tilde{\xi}_1)(\tilde{c}_0^2 + \tilde{\xi}_1)}}. \tag{4-32}$$

To eliminate the surface parameter from the above equation, we evaluate (4-30) on $\tilde{S}_1$ and substitute into the above equation

$$\alpha = \frac{\tilde{a}_0^3}{(\tilde{a}_0^2 + \tilde{\xi}_1)^{\frac{3}{2}}} \frac{\tilde{e}_0\tilde{E}_0}{\tilde{e}_1\tilde{E}_1}. \tag{4-33}$$

Moreover, we evaluate (4-30) on $\tilde{S}_0$ and $\tilde{S}_1$ to give

$$\tilde{e}_0 = \frac{\tilde{c}_0}{\tilde{a}_0}, \qquad \tilde{e}_1^2 = \sqrt{\frac{\tilde{c}_0^2 + \tilde{\xi}_1}{\tilde{a}_0^2 + \tilde{\xi}_1}}. \tag{4-34}$$

Eliminating $\tilde{c}_0$ from the above relations, we obtain

$$\tilde{\xi}_1 = \tilde{a}_0^2 \frac{\tilde{e}_1^2 - \tilde{e}_0^2}{1 - \tilde{e}_0^2}. \tag{4-35}$$

Hence, substitution into (4-33) yields

$$\alpha = \left(\frac{\tilde{e}_1^2 - \tilde{e}_0^2}{1 - \tilde{e}_0^2}\right)^{\frac{3}{2}} \frac{\tilde{e}_0\tilde{E}_0}{\tilde{e}_1\tilde{E}_1}. \tag{4-36}$$

Finally, since

$$\tilde{E}_0 = \sqrt{1 - \tilde{\kappa}^2(1 - \tilde{e}_0^2)}, \qquad \tilde{E}_1 = \sqrt{1 - \tilde{\kappa}^2(1 - \tilde{e}_1^2)}, \tag{4-37}$$

we note that

$$\tilde{e}_0 = \varrho_s e_0, \qquad \tilde{e}_1 = \varrho_s e_1, \tag{4-38}$$

Hence the volume ratio of the interior ellipsoid in terms of eccentricities may be written as follows

$$\alpha = \frac{(1 - \varrho_s^2 e_1^2)^{\frac{3}{2}} / e_1 \sqrt{1 - \tilde{\kappa}^2(1 - \varrho_s^2 e_1^2)}}{(1 - \varrho_s^2 e_0^2)^{\frac{3}{2}} / e_0 \sqrt{1 - \tilde{\kappa}^2(1 - \varrho_s^2 e_0^2)}} \tag{4-39}$$

Fig. [4-2] depicts the relationships between eccentricities and the volume ratio between composite ellipsoids.

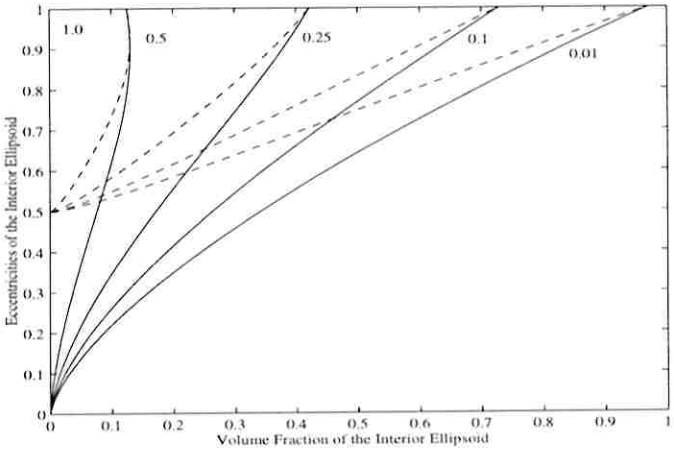



Figure 4-2: *Meridian and equatorial (dashed) eccentricities of an interior ellipsoid as functions of the volume fraction of the ellipsoid at various permeability anisotropy ratios. The curves refer to* $e_1 = 1.0$ *and* $\tilde{\kappa}^2 = 0.75$.

Before we proceed further, it may be shown that

$$-\left(\tilde{x}\frac{\partial \Gamma^{\tilde{x}}}{\partial \tilde{x}}\right)_{\tilde{\xi}} = -\left(\tilde{y}\frac{\partial \Gamma^{\tilde{y}}}{\partial \tilde{y}}\right)_{\tilde{\xi}} = -\left(\tilde{z}\frac{\partial \Gamma^{\tilde{z}}}{\partial \tilde{z}}\right)_{\tilde{\xi}} = \frac{2c_0}{\sqrt{(\tilde{a}_0^2 + \tilde{\xi})(\tilde{b}_0^2 + \tilde{\xi})(\tilde{c}_0^2 + \tilde{\xi})}} \qquad \text{for } 0 \leq \tilde{\xi} \leq \tilde{\xi}_1. \tag{4-40}$$

Evaluating the above equations on $\tilde{S}_0$, we put

$$c_0 = \frac{1}{2}\tilde{a}_0 \tilde{b}_0 \tilde{c}_0, \tag{4-41}$$

such that

$$\left(\tilde{x}\frac{\partial \Gamma^{\tilde{x}}}{\partial \tilde{x}}\right)_{\tilde{\xi}=0} = \left(\tilde{y}\frac{\partial \Gamma^{\tilde{y}}}{\partial \tilde{y}}\right)_{\tilde{\xi}=0} = \left(\tilde{z}\frac{\partial \Gamma^{\tilde{z}}}{\partial \tilde{z}}\right)_{\tilde{\xi}=0} = -1. \tag{4-42}$$

Evaluating (4-40) on $\tilde{S}_1$, we obtain

$$-\left(\tilde{x}\frac{\partial\Gamma^{\tilde{x}}}{\partial\tilde{x}}\right)_{\tilde{\xi}_1} = -\left(\tilde{y}\frac{\partial\Gamma^{\tilde{y}}}{\partial\tilde{y}}\right)_{\tilde{\xi}_1} = -\left(\tilde{z}\frac{\partial\Gamma^{\tilde{z}}}{\partial\tilde{z}}\right)_{\tilde{\xi}_1} = \frac{\tilde{a}_0\tilde{b}_0\tilde{c}_0}{\sqrt{(\tilde{a}_0^2+\tilde{\xi}_1)(\tilde{b}_0^2+\tilde{\xi}_1)(\tilde{c}_0^2+\tilde{\xi}_1)}}, \tag{4-43}$$

which is the volume ratio of the interior ellipsoid relative to the volume of the exterior ellipsoid. Hence

$$\left(\tilde{x}\frac{\partial\Gamma^{\tilde{x}}}{\partial\tilde{x}}\right)_{\tilde{\xi}_1} = \left(\tilde{y}\frac{\partial\Gamma^{\tilde{y}}}{\partial\tilde{y}}\right)_{\tilde{\xi}_1} = \left(\tilde{z}\frac{\partial\Gamma^{\tilde{z}}}{\partial\tilde{z}}\right)_{\tilde{\xi}_1} = -\alpha, \tag{4-44}$$

## 4.5 Continuity Conditions on the Surfaces of the Composite Inclusion

We are now in a position to apply the continuity conditions (4-5, 4-6) and (4-7, 4-8). The algebraic details are lengthy but straightforward and similar to those in Section 2.4. For instance, let $P_0(x, y, z) \in S_0$ and $P_1(x, y, z) \in S_1$ be arbitrary surface points. The conditions of the continuity of potentials require

$$(\boldsymbol{u}^T\boldsymbol{U} - \boldsymbol{v}_1^T\boldsymbol{V} - \boldsymbol{v}_2^T\boldsymbol{\Gamma}(\varrho_s e_0)\boldsymbol{V})\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{P_0} = 0. \tag{4-45}$$

$$(\boldsymbol{v}_1^T\boldsymbol{V} + \boldsymbol{v}_2^T\boldsymbol{\Gamma}(\varrho_s e_1)\boldsymbol{V} - \boldsymbol{w}^T\boldsymbol{\Gamma}(\varrho e_1)\boldsymbol{W} + \boldsymbol{J}^T)\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{P_1} = 0. \tag{4-46}$$

To satisfy these equations, we must have

$$\boldsymbol{u}^T\boldsymbol{U} - \boldsymbol{v}_1^T\boldsymbol{V} - \boldsymbol{v}_2^T\boldsymbol{\Gamma}(\varrho_s e_0)\boldsymbol{V} = \boldsymbol{0}, \tag{4-47}$$

$$\boldsymbol{v}_1^T\boldsymbol{V} + \boldsymbol{v}_2^T\boldsymbol{\Gamma}(\varrho_s e_1)\boldsymbol{V} - \boldsymbol{w}^T\boldsymbol{\Gamma}(\varrho e_1)\boldsymbol{W} + \boldsymbol{J}^T = \boldsymbol{0}. \tag{4-48}$$

For the continuity of the normal fluxes, let $f_0^-$ and $f_0^+$ be the left- and right-hand side of (4-6) i.e.,

$$f_0^- = k_x^i\frac{\partial\Phi^i}{\partial x}n_x + k_y^i\frac{\partial\Phi^i}{\partial y}n_y + k_z^i\frac{\partial\Phi^i}{\partial z}n_z, \tag{4-49}$$

$$f_0^+ = k_x^s\frac{\partial\Phi^s}{\partial x}n_x + k_y^s\frac{\partial\Phi^s}{\partial y}n_y + k_z^s\frac{\partial\Phi^s}{\partial z}n_z. \tag{4-50}$$

Similarly, for the left- and right-hand side of (4-8), we define

$$f_1^- = k_x^s\frac{\partial\Phi^s}{\partial x}m_x + k_y^s\frac{\partial\Phi^s}{\partial y}m_y + k_z^s\frac{\partial\Phi^s}{\partial z}m_z. \tag{4-51}$$

$$f_1^+ = k_x\frac{\partial\Phi^m}{\partial x}m_x + k_y\frac{\partial\Phi^m}{\partial y}m_y + k_z\frac{\partial\Phi^m}{\partial z}m_z. \tag{4-52}$$

where the outward unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ are given by

$$\boldsymbol{n} = \varepsilon_0 \boldsymbol{D}_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad \text{where } \frac{1}{\varepsilon_0} = \frac{1}{2}\|\nabla(\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D}_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1)\| \qquad (x, y, z) \in S_0, \tag{4-53}$$

$$\boldsymbol{m} = \varepsilon_1 \boldsymbol{D}_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{where } \frac{1}{\varepsilon_1} = \frac{1}{2}\|\nabla(\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \boldsymbol{D}_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1)\| \qquad (x, y, z) \in S_1. \tag{4-54}$$

After brief algebraic manipulations, we find that

$$f_0^- = \varepsilon_0 \boldsymbol{u}^T \boldsymbol{k}^i \boldsymbol{U} \boldsymbol{D}_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_0, \tag{4-55}$$

$$f_0^+ = \varepsilon_0 (\boldsymbol{v}_1^T \boldsymbol{k}^s \boldsymbol{V} + \boldsymbol{v}_2^T \boldsymbol{k}^s (\boldsymbol{\Gamma}(\varrho_s e_0) - \boldsymbol{I})\boldsymbol{V}) \boldsymbol{D}_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_0. \tag{4-56}$$

$$f_1^- = \varepsilon_1 (\boldsymbol{v}_1^T \boldsymbol{k}^s \boldsymbol{V} + \boldsymbol{v}_2^T \boldsymbol{k}^s (\boldsymbol{\Gamma}(\varrho_s e_1) - \alpha \boldsymbol{I})\boldsymbol{V}) \boldsymbol{D}_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_1, \tag{4-57}$$

$$f_1^+ = \varepsilon_1 (\boldsymbol{w}^T \boldsymbol{k}^m \boldsymbol{W} + \boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_1) - \boldsymbol{I})\boldsymbol{W}) \boldsymbol{D}_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (x, y, z) \in S_1. \tag{4-58}$$

Since we require

$$f_0^- = f_0^+ \qquad \text{for } P_0(x, y, z) \in S_0, \qquad f_1^- = f_1^+ \qquad \text{for } P_1(x, y, z) \in S_0 \tag{4-59}$$

the conditions of the continuity of the normal fluxes lead to

$$\boldsymbol{u}^T \boldsymbol{k}^i \boldsymbol{U} - \boldsymbol{v}_1^T \boldsymbol{k}^s \boldsymbol{V} - \boldsymbol{v}_2^T \boldsymbol{k}^s \boldsymbol{V}(\boldsymbol{\Gamma}(\varrho_s e_0) - \boldsymbol{I}) = \boldsymbol{0}, \tag{4-60}$$

$$\boldsymbol{v}_1^T \boldsymbol{k}^s \boldsymbol{V} + \boldsymbol{v}_2^T \boldsymbol{k}^s (\boldsymbol{\Gamma}(\varrho_s e_1) - \alpha \boldsymbol{I})\boldsymbol{V} - \boldsymbol{w}^T \boldsymbol{k}^m (\boldsymbol{\Gamma}(\varrho e_1) - \boldsymbol{I})\boldsymbol{W} + \boldsymbol{J}^T \boldsymbol{k}^m = \boldsymbol{0}. \tag{4-61}$$

Finally, by transposing the equations in (4-47), (4-48), (4-60) and (4-61), the vectors $\boldsymbol{u}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{w}$ are simultaneously determined by solving the following matrix equation

$$\boldsymbol{\Pi} \chi = \boldsymbol{\Upsilon} \tag{4-62}$$

where

$$\chi = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{w} \end{bmatrix}; \qquad \boldsymbol{\Upsilon} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -\boldsymbol{J} \\ -\boldsymbol{k}\boldsymbol{J} \end{bmatrix}, \tag{4-63}$$

and

$$
\mathbf{\Pi} = \begin{bmatrix} \boldsymbol{U} & -\boldsymbol{V} & -\boldsymbol{V}\boldsymbol{\Gamma}(\varrho_s e_0|\tilde{\kappa}) & \mathbf{0} \\ \boldsymbol{k}^i\boldsymbol{U} & -\boldsymbol{k}^s\boldsymbol{V} & -\boldsymbol{k}^s\boldsymbol{V}\boldsymbol{\Gamma}(\varrho_s e_0|\tilde{\kappa}) - \boldsymbol{I}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{V} & \boldsymbol{V}\boldsymbol{\Gamma}(\varrho_s e_1|\tilde{\kappa}) & -\boldsymbol{W}\boldsymbol{\Gamma}(\varrho e_1|\acute{\kappa}) \\ \mathbf{0} & \boldsymbol{k}^s\boldsymbol{V} & \boldsymbol{k}^s\boldsymbol{V}(\boldsymbol{\Gamma}(\varrho_s e_1|\tilde{\kappa}) - \alpha\boldsymbol{I}) & -\boldsymbol{k}^m\boldsymbol{W}(\boldsymbol{\Gamma}(\varrho e_1|\acute{\kappa}) - \boldsymbol{I}) \end{bmatrix} \qquad (4\text{-}64)
$$

For the cases of spheroidal inclusions, the surface values of associated external harmonics in the above matrix is replaces by

$$
\boldsymbol{\Gamma}(s|0) = \begin{bmatrix} \frac{1}{2}\lambda(s|0) & & \\ & \frac{1}{2}\lambda(s|0) & \\ & & 1 - \lambda(s|0) \end{bmatrix} \qquad s = \varrho_s e_0, \varrho_s e_1, \varrho e_1. \qquad (4\text{-}65)
$$

for oblate spheroidal inclusion, where $\lambda(,|0)$ is defined in (2-89). For prolate spheroidal inclusion,

$$
\boldsymbol{\Gamma}(s|1) = \begin{bmatrix} 1 - 2\mu(s|1) & & \\ & \mu(s|1) & \\ & & \mu(s|1) \end{bmatrix} \qquad s = \varrho_s e_0, \varrho_s e_1, \varrho e_1. \qquad (4\text{-}66)
$$

Here $\mu(,|1)$ is the surface value of (2-86). Accordingly, the volume ratio of the interior spheroid becomes

$$
\alpha = \frac{(1 - \varrho_s^2 e_1^2)^{\frac{3}{2}}/e_1}{(1 - \varrho_s^2 e_0^2)^{\frac{3}{2}}/e_0} \qquad \text{(Oblate spheroids).} \qquad (4\text{-}67)
$$

$$
\alpha = \frac{(1 - \varrho_s^2 e_1^2)^{\frac{3}{2}}/e_1^2}{(1 - \varrho_s^2 e_0^2)^{\frac{3}{2}}/e_0^2} \qquad \text{(Prolate spheroids).} \qquad (4\text{-}68)
$$

# Examples & Figures

The analytical solutions developed in the preceding chapters give generalized solutions for the distribution of pressure inside and outside an ellipsoidal inclusion over a wide range of inclusion geometry, permeability orientation and anisotropy ratios. For the purpose of illustration, we finally present here seven examples to be compared with a base case whose input data is given by the following Table [E-1]. Observe that we now explicitly express the rotation matrices in (3-16) and (3-19) as functions of the 3D rotation angles (see Appendix B).

| Parameter | Symbol | Value |
|---|---|---|
| Meridian eccentricity | $e_0$ | 1/10 |
| Equatorial eccentricity | $E_0$ | 1/2 |
| Prescribed potential gradient | $\boldsymbol{J}$ | (1/10, 1/10, 1) |
| Permeability contrast | $\boldsymbol{\delta}$ | (1/2, 1/2, 1/10) |
| Rotation Matrices | $\boldsymbol{Q}(\phi^m, \theta^m, \psi^m)$ | diag(1, 1, 1) |
| | $\boldsymbol{Q}_i(\phi^i, \theta^i, \psi^i)$ | diag(1, 1, 1) |
| Matrix Anisotropy ratio | $k_z^m/k_x^m$ | 1/10 |
| | $k_y^m/k_x^m$ | 4/5 |
| Inclusion Anisotropy ratio | $k_z^i/k_x^i$ | 1/20 |
| | $k_y^i/k_x^i$ | 1 |
| Matrix permeability in $x$-direction | $k_x^m$ | 50 |

Table [E-1]: *Base Case data*

The results are presented by contour plots[1] of the pressure distribution in the $xz$-plane. In all figures, the distances in $x$ and $z$ are scaled i.e, $x_D = x/a$ and $z_D = z/c$.

- **Example I**: In this example we study the effect of permeability contrast on the distribution of pressure by first making the contrast tensor $\boldsymbol{\delta}$ approach zero. In this case, the distribution of pressure is given by the equations in (2-147 & 2-148). Next, we let the contrast tensor $\boldsymbol{\delta} \to \infty$. Hence the internal potential is constant and zero, refer to equations in (2-151). Finally, we let $\boldsymbol{\delta} \to \boldsymbol{I}$ and find that the pressure distribution is uniform, as it should be, see equations in (2-152).

  Contour plots for the distribution of potentials are given in Fig. [E-2].

[1]The numerical computation and the graphical results are made by a *Mathematica*[20] program. It is therefore emphasized that the blemishes in the contours around the ellipsoid are due to the *Mathematica* algorithm that finds contour lines. This algorithm is found to be susceptible to aliasing which causes such misleading breaks or turns in the contour lines[21].

- **Example II**: We now consider two special cases of the inclusion problem reported in [1] and [7]. First, we consider the distribution of pressure inside and outside an oblate spheroid of isotropic permeability $k^i$ and we let $k_x^m = k_y^m$ for the matrix surrounding the inclusion. This example illustrates the distribution of internal and external potentials given in (2-153 & 2-154). In the other case, we consider isotropic ellipsoidal inclusion in an isotropic matrix of infinite dimension. The solutions are given by equations in (2-156 & 2-157).

  The contour plots in Fig. [E-3] illustrate the distribution of potentials.

- **Example III**: By keeping everything else unchanged, we reduce the permeability anisotropy of the inclusion and the surrounding medium by a factor of 10. Then we increase the anisotropy ratio by a factor of 10. Contour plots of potential distribution are given in Fig. [E-4].

- **Example IV**: We now study, again, the effect of permeability contrast by first reducing the contrast tensor with a factor of 10. Then we increase the the contrast with a factor of 10. The graphical results are given in Fig. [E-5].

- **Example V**: In the preceding examples, the meridian and equatorial eccentricity were kept at 0.1 and 0.5. We now vary the geometry of the inclusion by first making $e_0$ approach zero while leaving $E_0$ unchanged. In the other case, we let $e_0 \to 1$ and, again with fixed $E_0$. The distributions of pressure for the above cases are given in Fig. [E-6].

- **Example VI**: In this example, we let the permeability of the inclusion and the surrounding matrix be arbitrarily oriented with respect to the principal axes of the ellipsoid. We consider two cases in which we first let an arbitrary orientation of the inclusion permeability tensor while keeping the permeability tensor of the matrix aligned with the principal axes of the ellipsoid. The orientation of the inclusion permeability tensor is described by the rotation matrix $\boldsymbol{Q}_i$ whose entries are now determined by making a successive 3D axes rotation (that leaves the origin unchanged) in terms of the Euler angles $(\phi^i, \theta^i, \psi^i) = (\pi/2, \pi/4, 3\pi/2)$, see Appendix B.

  In the other case, we let the permeability tensor of the medium surrounding the inclusion be arbitrarily oriented relative to the axes of the ellipsoid while keeping the permeability tensor of the inclusion aligned with principal axes of the ellipsoid. The rotation angles are $(\phi^m, \theta^m, \psi^m) = (\pi/2, \pi/4, 3\pi/2)$.

  The contour plots in Fig. [E-7] depict the distribution of the internal and external pressures for the above cases.

- **Example VII**: In this final example, we let both permeability tensors be arbitrarily oriented relative to the axes of the ellipsoid. First, we let the permeability tensors have identical arbitrary orientation described by the Euler angles $(\pi/2, \pi/4, 3\pi/2)$. Then we differ the the arbitrary orientation of $\boldsymbol{k}^i$ from the orientation of $\boldsymbol{k}^m$ by making $(\phi^i, \theta^i, \psi^i) = (3\pi/2, \pi/4, \pi/2)$.

  Fig. [E-8] shows the distributions of internal and external potentials with the above cases of permeability orientations.

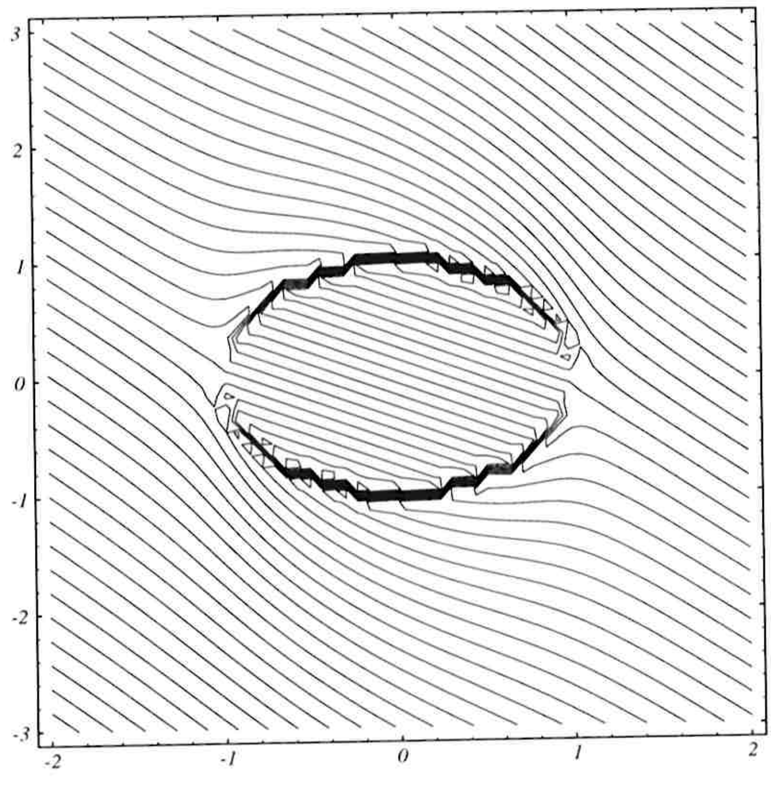



Figure **E-1**: Distribution of Base Case potential in the scaled $(x_D, z_D)$-plane.

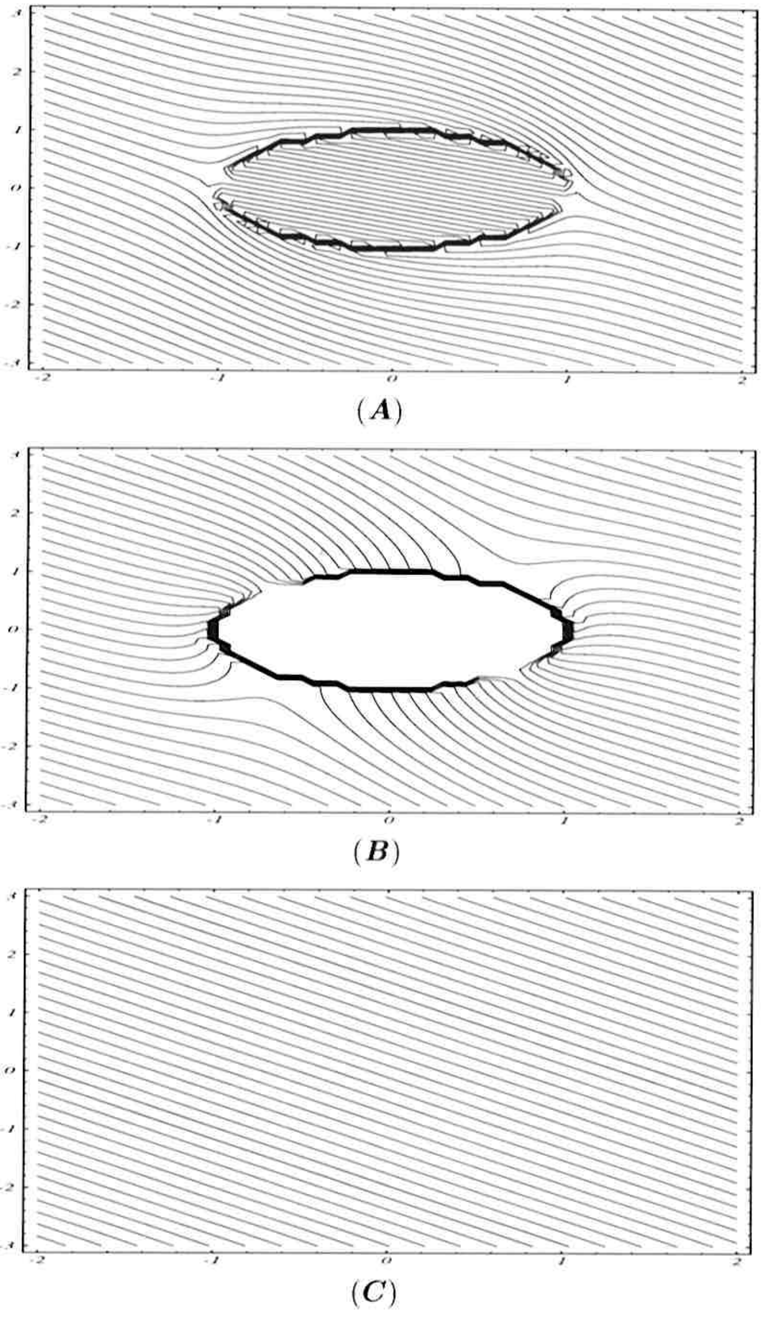
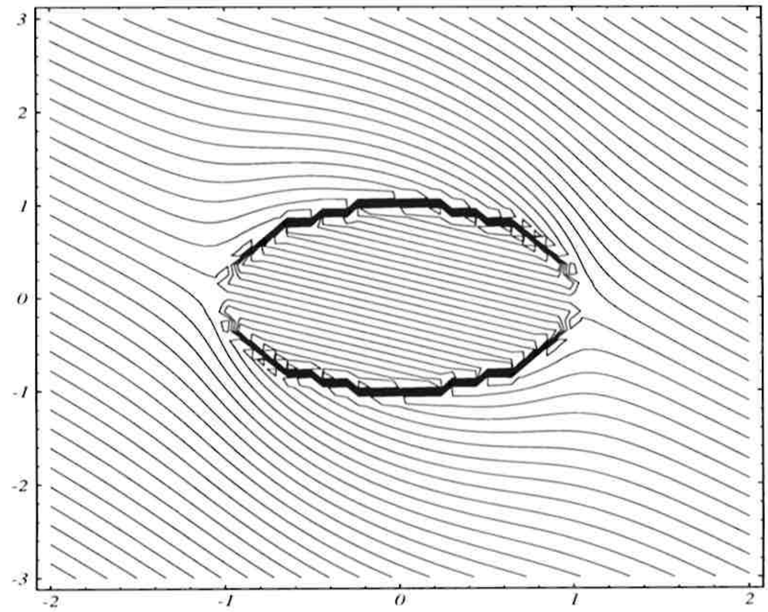
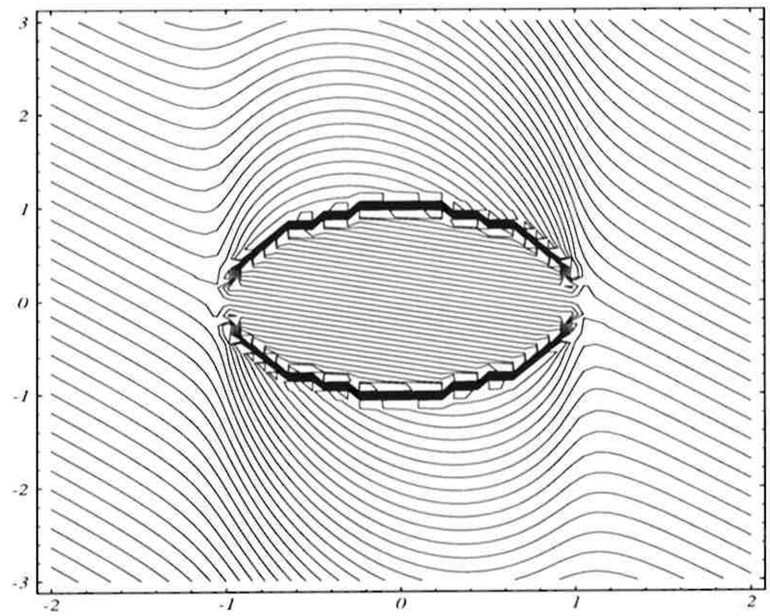



Figure **E-2**: Distribution of potential in the $(x_D, z_D)$-plane for **Example I**: (A) $\boldsymbol{\delta \to 0}$, (B) $\boldsymbol{\delta} \to \infty$, (C) $\boldsymbol{\delta \to I}$.




($\boldsymbol{A}$)




($\boldsymbol{B}$)

Figure **E-3**: Distribution of potential in the $(x_D, z_D)$-plane for **Example II**: (A) Inclusion of isotropic oblate spheroid in an anisotropic matrix for $\boldsymbol{\delta} = k^i \boldsymbol{k}^m$. (B) Both the inclusion and the surrounding medium are isotropic i.e., $\boldsymbol{\delta} = (k^i/k^m)\boldsymbol{I}$.

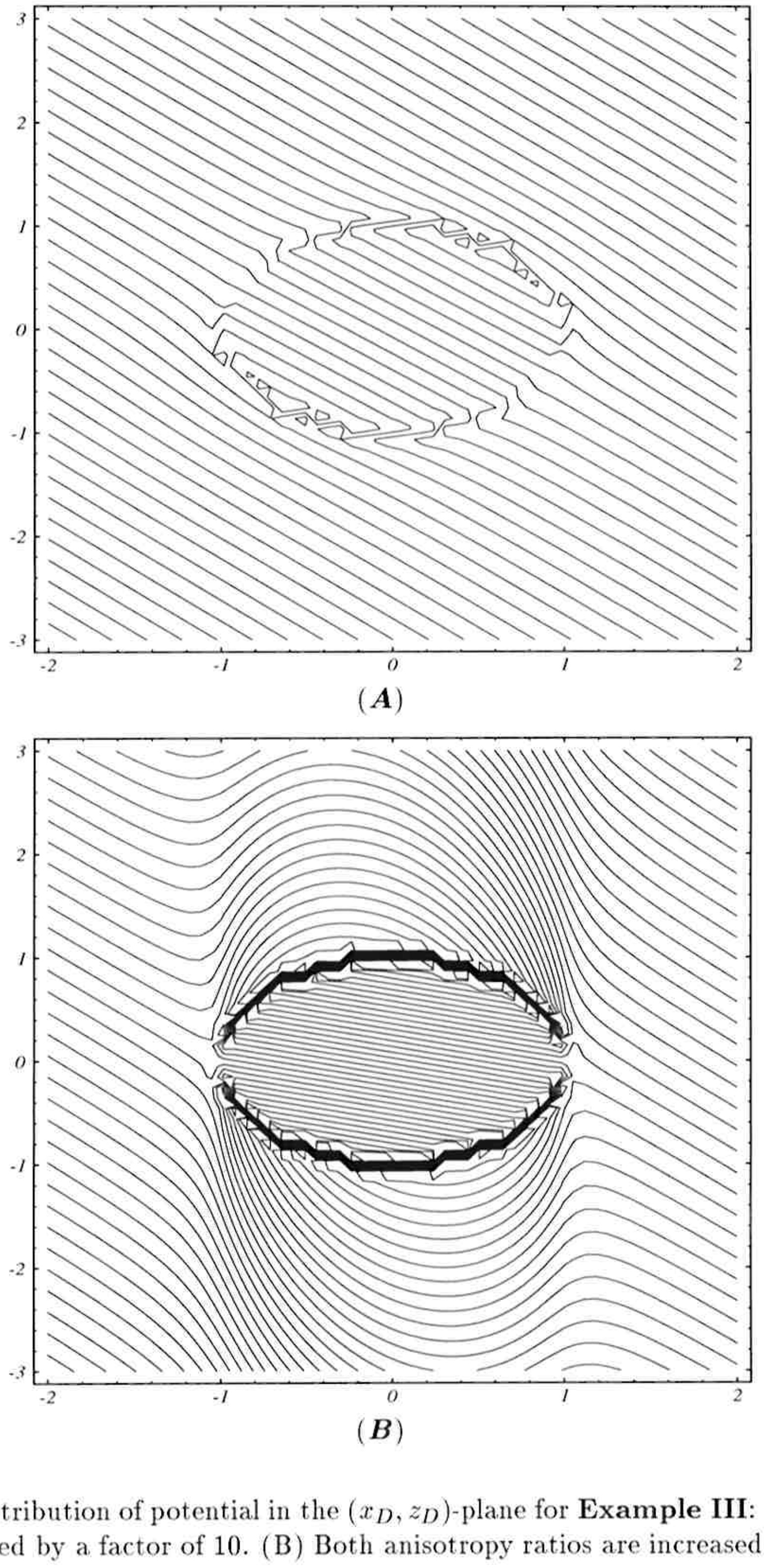
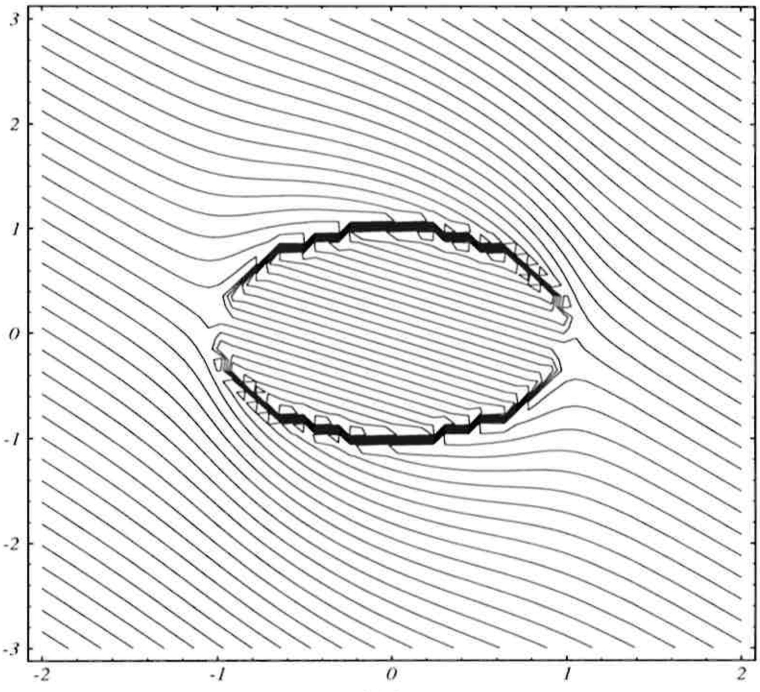
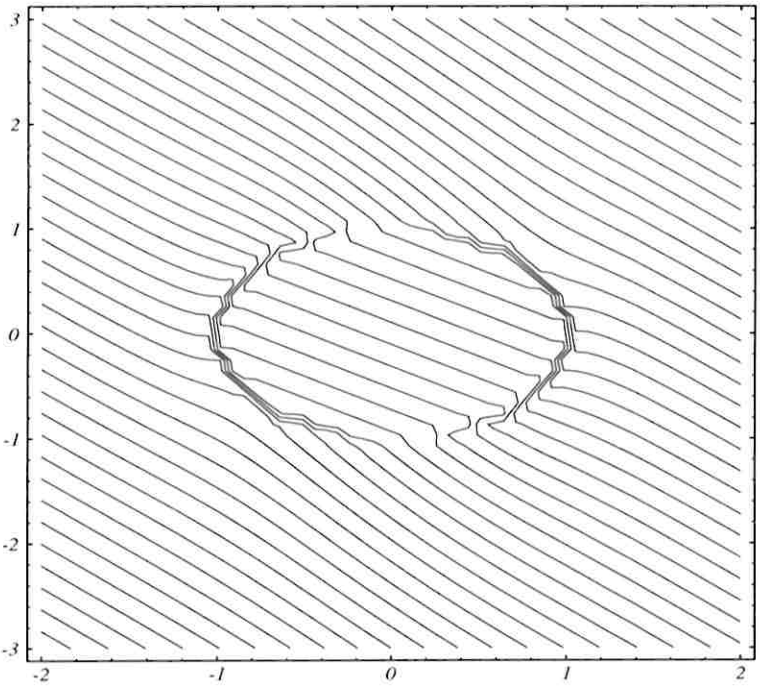



Figure **E-4**: Distribution of potential in the $(x_D, z_D)$-plane for **Example III**: (A) $k_z^m/k_x^m$ and $k_z^i/k_x^i$ are reduced by a factor of 10. (B) Both anisotropy ratios are increased by a factor 10.




(**A**)




(**B**)

Figure **E-5**: Distribution of potential in $(x_D, z_D)$-plane for **Example IV**: (A) Permeability contrast is reduced by a factor of 10. (B) Increased by a factor of 10.

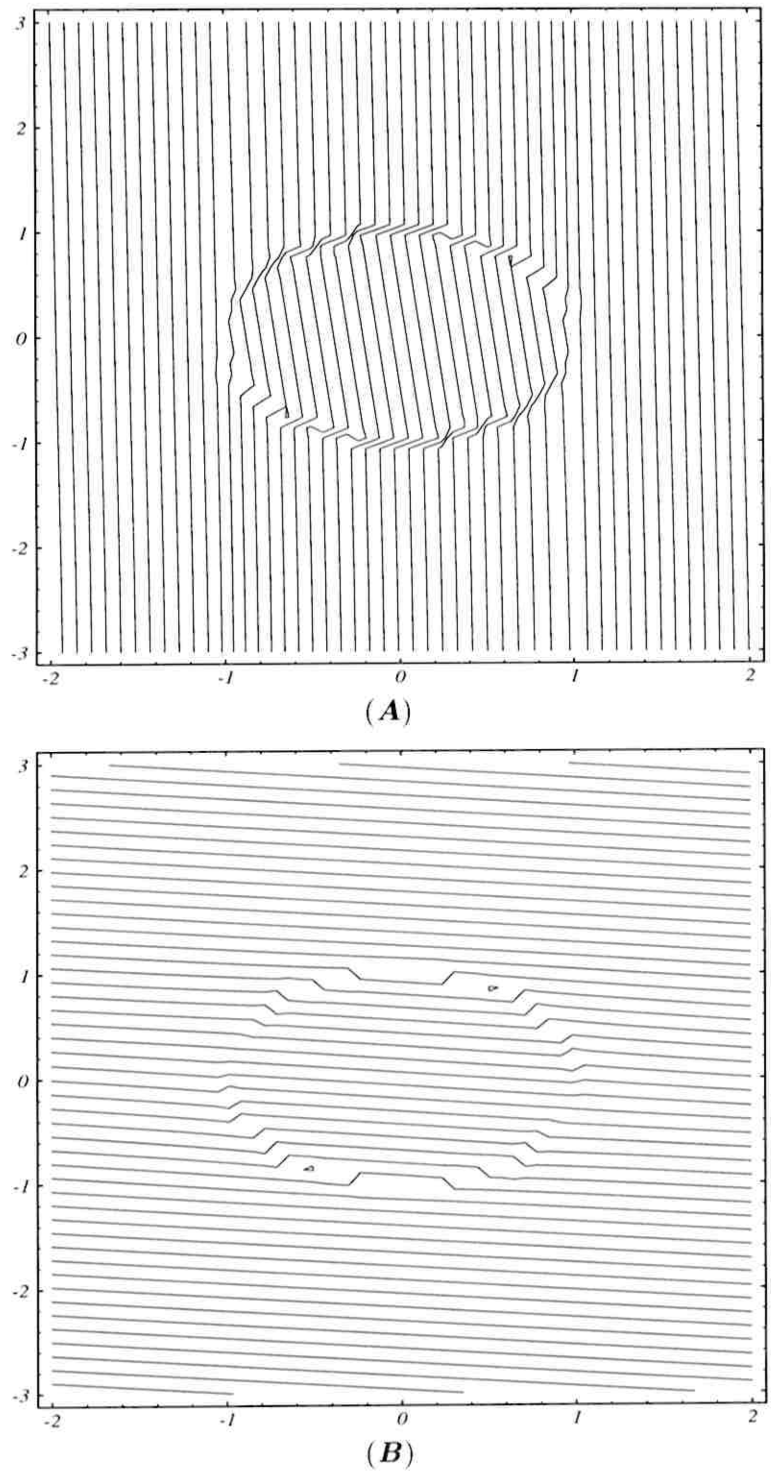
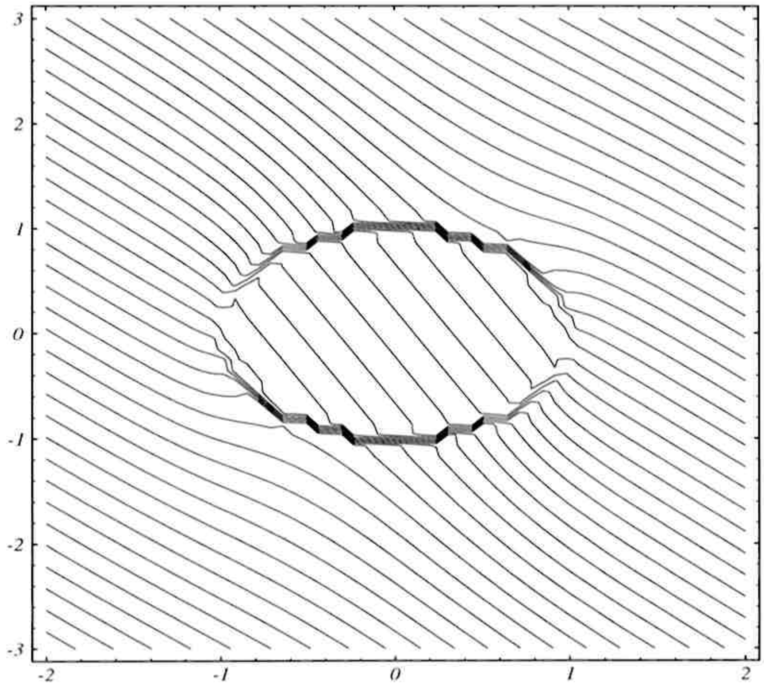
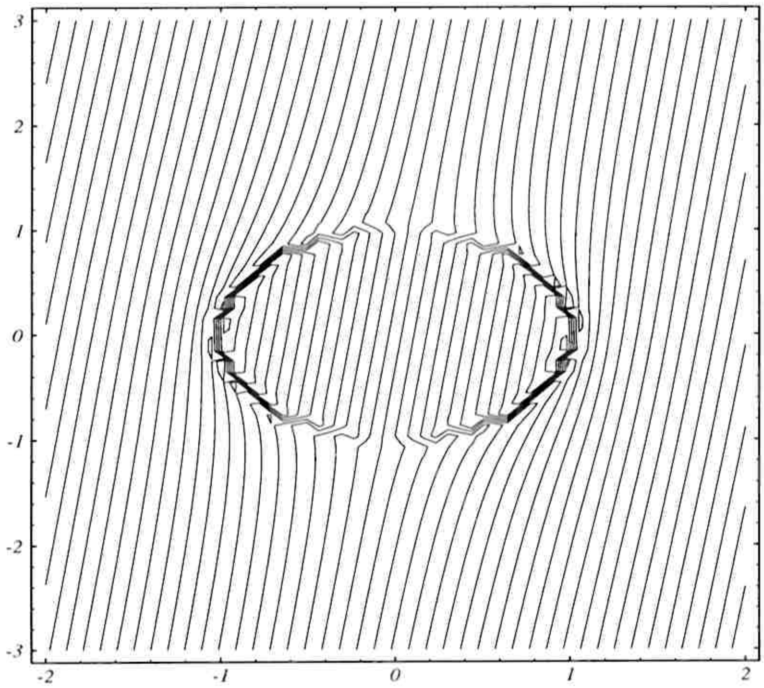



Figure **E-6**: Distribution of potential in $(x_D, z_D)$-plane for **Example V**: (A) $e_0 \to 0$ (B) $e_0 \to 1$.

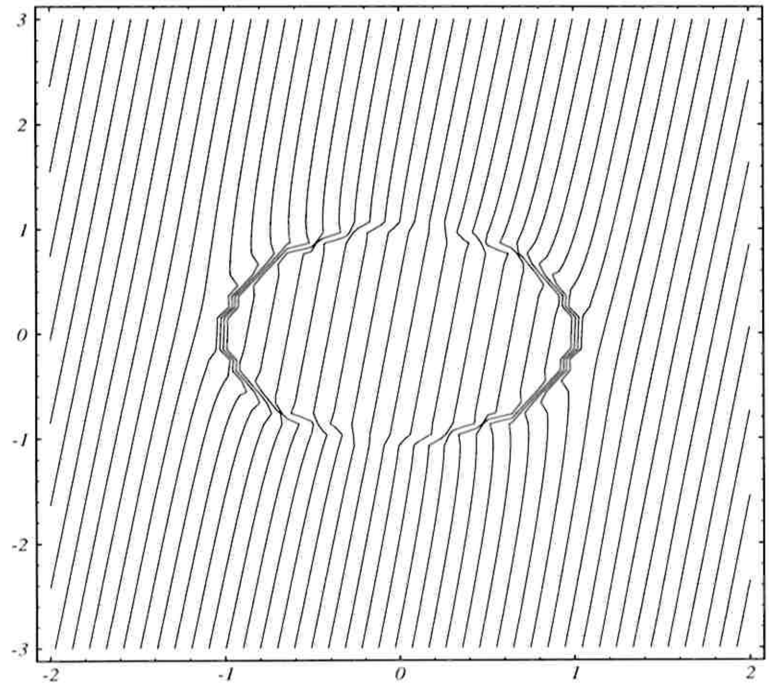



**(A)**

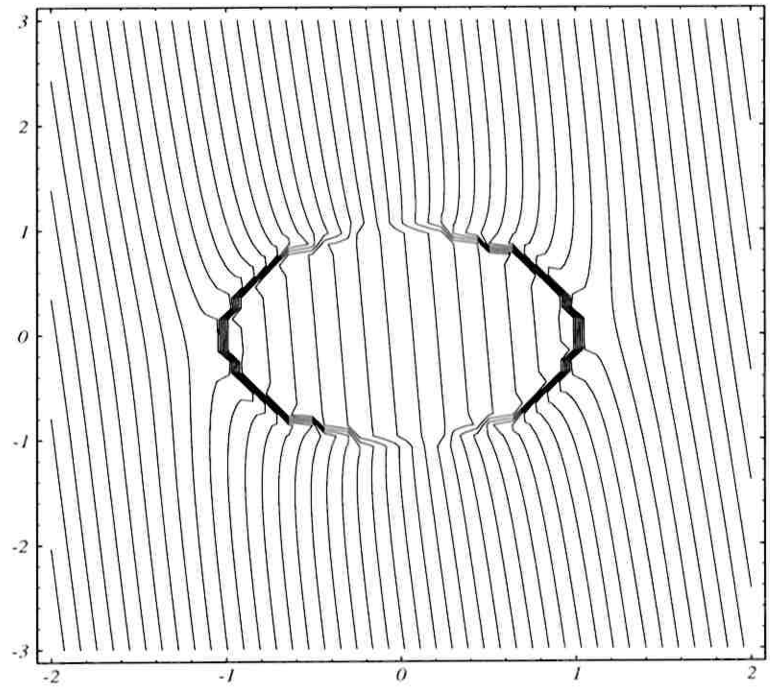



**(B)**

Figure **E-7**: Distribution of potential in $(x_D, z_D)$-plane for **Example VI**: (A) Orientation of inclusion permeability tensor - $\boldsymbol{Q}_i(\pi/2, \pi/4, 3\pi/2)$, (B) Orientation of matrix permeability tensor - $\boldsymbol{Q}(\pi/2, \pi/4, 3\pi/2)$.




(*A*)




(*B*)

Figure **E-8**: Distribution of potential in $(x_D, z_D)$-plane for **Example VII**: Arbitrary orientation of permeability tensors - (A) $\boldsymbol{Q} = \boldsymbol{Q}_i(\pi/2, \pi/4, 3\pi/2)$, (B) $\boldsymbol{Q}(\pi/2, \pi/4, 3\pi/2)$, $\boldsymbol{Q}_i(3\pi/2, \pi/4, \pi/2)$

# Laplace's Equation in Ellipsoidal Coordinates

For a point $P(x, y, z)$ in space we may regard $x$, $y$ and $z$ as functions of $\xi$, $\eta$, $\zeta$

$$x = x(\xi, \eta, \zeta), \qquad y = y(\xi, \eta, \zeta), \qquad z = z(\xi, \eta, \zeta), \tag{A-1}$$

such that if values are assigned to $\xi, \eta, \zeta$, then the intersection of the corresponding surfaces - one from each family - gives the coordinates of the point. Thus, denoting by $\boldsymbol{s}$ a position vector for the the distance between a point $P$ in $\Omega_1$ whose coordinates are uniquely determined by the intersection $(\xi, \eta, \zeta)$ and an arbitrary neighbouring point $Q$ whose coordinates are fixed by the intersection of the family members $\xi + d\xi =$, $\eta + d\eta =$, $\zeta + d\zeta =$const., consider the element of length in ellipsoidal coordinates

$$d\boldsymbol{s} = \frac{\partial \boldsymbol{s}}{\partial \xi} d\xi + \frac{\partial \boldsymbol{s}}{\partial \eta} d\eta + \frac{\partial \boldsymbol{s}}{\partial \zeta} d\zeta, \tag{A-2}$$

Accordingly,

$$\begin{aligned} ds^2 = \frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \xi} d\xi^2 &+ \frac{\partial \boldsymbol{s}}{\partial \eta}\frac{\partial \boldsymbol{s}}{\partial \eta} d\eta^2 + \frac{\partial \boldsymbol{s}}{\partial \zeta}\frac{\partial \boldsymbol{s}}{\partial \zeta} d\zeta^2 \\ &+ 2\frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \eta} d\xi d\eta + 2\frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \zeta} d\xi d\zeta + 2\frac{\partial \boldsymbol{s}}{\partial \eta}\frac{\partial \boldsymbol{s}}{\partial \zeta} d\eta d\zeta. \end{aligned} \tag{A-3}$$

Defining

$$h_\xi^2 = \frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \xi}; \qquad h_\eta^2 = \frac{\partial \boldsymbol{s}}{\partial \eta}\frac{\partial \boldsymbol{s}}{\partial \eta}; \qquad h_\zeta^2 = \frac{\partial \boldsymbol{s}}{\partial \zeta}\frac{\partial \boldsymbol{s}}{\partial \zeta}, \tag{A-4}$$

the unit vectors along the surface coordinates are

$$\boldsymbol{e}_\xi = \frac{1}{h_\xi}\frac{\partial \boldsymbol{s}}{\partial \xi}, \qquad \boldsymbol{e}_\eta = \frac{1}{h_\eta}\frac{\partial \boldsymbol{s}}{\partial \eta}, \qquad \boldsymbol{e}_\zeta = \frac{1}{h_\zeta}\frac{\partial \boldsymbol{s}}{\partial \zeta}, \tag{A-5}$$

Confining ourselves to orthogonal systems in which the surfaces $\xi, \eta, \zeta$ meet at right angles, we must have

$$\frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \eta} = \frac{\partial \boldsymbol{s}}{\partial \xi}\frac{\partial \boldsymbol{s}}{\partial \zeta} = \frac{\partial \boldsymbol{s}}{\partial \eta}\frac{\partial \boldsymbol{s}}{\partial \zeta} = 0 \tag{A-6}$$

Consequently,

$$ds^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\zeta^2 d\zeta^2. \tag{A-7}$$

Now, expressing the element of length in Cartesian coordinates,

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{A-8}$$

where,

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta, \tag{A-9}$$

the scaling factors of the curvilinear coordinates, $h_\xi, h_\eta, h_\zeta$ are determined by equating (A-7) and (A-8)

$$dx^2 + dy^2 + dz^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\zeta^2 d\zeta^2. \tag{A-10}$$

Thus,

$$\begin{aligned} h_\xi^2 &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2, \\ h_\eta^2 &= \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2, \\ h_\zeta^2 &= \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2, \end{aligned} \tag{A-11}$$

Now, the Laplacian in terms of scaling factors, here $h_\xi$, $h_\eta$, $h_\zeta$, is readily given by the formula (see Kreyszig, 1988):

$$\nabla^2 = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial \xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{h_\eta h_\xi}{h_\zeta}\frac{\partial}{\partial \zeta}\right)\right]. \tag{A-12}$$

where, by logarithmic derivation of the equations in (1-9) with respect to $\xi$, $\eta$, $\zeta$,

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{x}{2(a^2+\xi)}; & \frac{\partial y}{\partial \xi} &= \frac{y}{2(b^2+\xi)}; & \frac{\partial z}{\partial \xi} &= \frac{z}{2(c^2+\xi)}, \\ \frac{\partial x}{\partial \eta} &= \frac{x}{2(a^2+\eta)}; & \frac{\partial y}{\partial \eta} &= \frac{y}{2(b^2+\eta)}; & \frac{\partial z}{\partial \eta} &= \frac{z}{2(c^2+\eta)}, \\ \frac{\partial x}{\partial \zeta} &= \frac{x}{2(a^2+\zeta)}; & \frac{\partial y}{\partial \zeta} &= \frac{y}{2(b^2+\zeta)}; & \frac{\partial z}{\partial \zeta} &= \frac{z}{2(c^2+\zeta)}. \end{aligned} \tag{A-13}$$

Hence, substitution of the above expressions into (A-11) gives

$$\begin{aligned} h_\xi^2 &= \frac{1}{4}\frac{(\xi-\eta)(\xi-\zeta)}{(a^2+\xi)(b^2+\xi)(c^2+\xi)}, \\ h_\eta^2 &= \frac{1}{4}\frac{(\eta-\xi)(\eta-\zeta)}{(a^2+\eta)(b^2+\eta)(c^2+\eta)}, \\ h_\zeta^2 &= \frac{1}{4}\frac{(\zeta-\eta)(\zeta-\xi)}{(a^2+\zeta)(b^2+\zeta)(c^2+\zeta)}. \end{aligned} \tag{A-14}$$

Thus,

$$\nabla^2\Phi = (\eta-\zeta)D(\xi)\frac{\partial}{\partial \xi}\left(D(\xi)\frac{\partial \Phi}{\partial \xi}\right) + (\zeta-\xi)D(\eta)\frac{\partial}{\partial \eta}\left(D(\eta)\frac{\partial \Phi}{\partial \eta}\right) + (\xi-\eta)D(\zeta)\frac{\partial}{\partial \zeta}\left(D(\zeta)\frac{\partial \Phi}{\partial \zeta}\right) = 0 \tag{A-15}$$

where

$$D(\alpha) = \sqrt{(a^2 + \alpha)(b^2 + \alpha)(c^2 + \alpha)}\,, \qquad \alpha = \xi, \eta, \zeta. \tag{A-16}$$

In connection with the coordinate transformations required to solve the external problems in this study, the above equations can be readily utilized after a change of notation.

# Formulae for Principal Axis Transformation

Suppose that the inclusion is described by a coordinate system with $x$ and $y$ in the horizontal plane and $z$ vertically upward. Assuming that the axes of the ellipsoid are aligned with the $(x, y, z)$ coordinate system, we perform an orthogonal transformation that takes the initial coordinate system $(x, y, z)$ into coincidence with another Cartesian coordinate system, we make a sequence of three successive rotations in $R^3$ through the *Euler angles* $(\phi^m, \theta^m, \psi^m)$ by proceeding as follows:

- Rotate the initial system of axes by an angle $\phi^m$ counterclockwise about the the $z$ axis (leaving the $z$ axis unchanged) and label the resultant coordinate system $(x', y', z)$. The transformation is then given by

$$\begin{bmatrix} x' \\ y' \\ z \end{bmatrix} = \boldsymbol{Q}_\phi \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \boldsymbol{Q}_\phi = \begin{bmatrix} \cos\phi^m & \sin\phi^m & 0 \\ -\sin\phi^m & \cos\phi^m & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{B-1}$$

- Rotate the intermediate coordinates $(x', y', z)$ around the $x'$ axis counterclockwise by an angle $\theta^m$ to produce yet another intermediate coordinate system, say $(x', y'', z')$. This transformation is given by

$$\begin{bmatrix} x' \\ y'' \\ z' \end{bmatrix} = \boldsymbol{Q}_\theta \begin{bmatrix} x' \\ y' \\ z \end{bmatrix}, \qquad \boldsymbol{Q}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^m & \sin\theta^m \\ 0 & -\sin\theta^m & \cos\theta^m \end{bmatrix}. \tag{B-2}$$

- Finally, Rotate the coordinate system $(x', y'', z')$ about $y''$ counterclockwise by an angle $\psi^m$ to produce the final coordinate system of axes $(x'', y'', z'')$ given by

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \boldsymbol{Q}_\psi \begin{bmatrix} x' \\ y'' \\ z' \end{bmatrix}, \qquad \boldsymbol{Q}_\psi = \begin{bmatrix} \cos\psi^m & \sin\psi^m & 0 \\ -\sin\psi^m & \cos\psi^m & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{B-3}$$

Thus the complete 3D axis rotation is obtained as the triple product of the separate rotations

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{B-4}$$

where

$$\boldsymbol{Q} = \boldsymbol{Q}_\psi \boldsymbol{Q}_\theta \boldsymbol{Q}_\phi. \tag{B-5}$$

We may now let $(x'', y'', z'')$ be the coordinate system in which the permeability tensor of the exterior space of the formation is diagonal. Hence by replacing $(x'', y'', z'')$ with $(\rho, \mu, \nu)$, the relative orientation of the permeability tensor $\boldsymbol{k}^m$ with respect to the axes of the ellipsoid is given by

$$\boldsymbol{Q}(\phi^m, \theta^m, \psi^m) = [\boldsymbol{q}_x(\phi^m, \theta^m, \psi^m), \boldsymbol{q}_y(\phi^m, \theta^m, \psi^m), \boldsymbol{q}_z(\phi^m, \theta^m, \psi^m)], \tag{B-6}$$

where

$$\begin{aligned}
\boldsymbol{q}_x(\phi^m, \theta^m, \psi^m) &= \begin{bmatrix} l_x \\ m_x \\ n_x \end{bmatrix} = \begin{bmatrix} \cos\psi^m \cos\phi^m - \cos\theta^m \sin\phi^m \sin\psi^m \\ -\sin\psi^m \cos\phi^m - \cos\psi^m \sin\phi^m \cos\phi^m \\ \sin\psi^m \sin\theta^m \end{bmatrix}, \\
\boldsymbol{q}_y(\phi^m, \theta^m, \psi^m) &= \begin{bmatrix} l_y \\ m_y \\ n_y \end{bmatrix} = \begin{bmatrix} \cos\psi^m \cos\phi^m + \cos\theta^m \sin\phi^m \sin\psi^m \\ -\sin\psi^m \sin\phi^m + \cos\theta^m \sin\phi^m \cos\phi^m \\ -\sin\theta^m \sin\phi^m \end{bmatrix}, \\
\boldsymbol{q}_z(\phi^m, \theta^m, \psi^m) &= \begin{bmatrix} l_z \\ m_z \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\psi^m \sin\theta^m \\ \cos\psi^m \sin\theta^m \\ \cos\theta^m \end{bmatrix}.
\end{aligned} \tag{B-7}$$

An analogous transformation may be derived for the relative orientation of the permeability tensor of the inclusion with respect to the axes of the ellipsoid. Denoting by $\phi^i, \theta^i, \psi^i$ the rotation angles, coordinate transformation along the the preceding sequence of axis rotation gives

$$\boldsymbol{Q}_i(\phi^i, \theta^i, \psi^i) = [\boldsymbol{q}^i_x(\phi^i, \theta^i, \psi^i), \boldsymbol{q}^i_y(\phi^i, \theta^i, \psi^i), \boldsymbol{q}^i_z(\phi^i, \theta^i, \psi^i)], \tag{B-8}$$

where $\boldsymbol{q}^i_x$, $\boldsymbol{q}^i_y$, $\boldsymbol{q}^i_z$ are given by replacing the Euler angles $\phi^m, \theta^m, \psi^m$ in ( B-7) with $\phi^i, \theta^i, \psi^i$.

# Acknowledgement

The material presented here is prepared for RF-Rogaland Research, Stavanger. Thanks are due to Dr. Steinar Ekrann for his useful comments, constructive criticism and suggestions.

# Nomenclature

## Symbols

| | | |
|---|---|---|
| $\alpha$ | - | Surface parameter; volume ratio of interior ellipsoid. |
| $\boldsymbol{k}$ | - | Permeability tensor. |
| $\Phi$ | - | Potential. |
| $\boldsymbol{r}$ | - | Position vector. |
| $\boldsymbol{J}$ | - | Constant potential gradient. |
| $a, b, c$ | - | Semiaxes of an ellipsoid . |
| $S$ | - | Ellipsoidal surface. |
| $n$ | - | outward normal . |
| $x, y, z$ | - | Coordinate system in which the ellipsoid is defined. |
| $\xi, \eta, \zeta$ | - | Ellipsoidal coordinates corresponding to $x, y, z$. |
| $h_\xi, h_\eta, h_\zeta$ | - | Scaling factors for the curvilinear cooridnates $\xi, \eta, \zeta$. |
| $\alpha, \beta, \gamma$ | - | Coordinate system aligned with principal directions of inclusion permeability. |
| $\rho, \mu, \nu$ | - | Coordinate system aligned with principal direction of matrix permeability. |
| $\tilde{x}, \tilde{y}, \tilde{z}$ | - | Coordinate system in which inclusion (or skin) permeability is isotropic . |
| $\tilde{p}, \tilde{q}$ | - | parameters defined in (2-15). |
| $\tilde{\lambda}$ | - | Ellipsoidal surface parameter for $-c^2 < \xi \leq 0$. |
| $\acute{x}, \acute{y}, \acute{z}$ | - | Coordinate system in which matrix permeability is isotropic . |
| $\acute{\xi}, \acute{\eta}, \acute{\zeta}$ | - | Ellipsoidal coordinates corresponding to $\acute{x}, \acute{y}, \acute{z}$. |
| $\phi, \theta, \psi$ | - | Euler's angles for 3D axis rotation. |
| $\Gamma$ | - | Associated external harmonics. |
| $\boldsymbol{e}_r$ | - | Unit vectors along coordinate directions $r = (x, y, z), (\rho, \mu, \nu)$ |
| $(l_r, m_r, n_r)$ | - | Direction cosines $r = (x, y, z)$ |
| $\chi$ | - | Function defined in text. |
| $e, E$ | - | Meridian and equatorial eccentricities. |
| $\kappa$ | - | Modulus of ellipsoidal eccentricities. |
| $\lambda(,\|,), \mu(,\|,)$ | - | Functions defining associated external harmonics for spheroids. |
| $F, E$ | - | Elliptic integrals of the 1st and 2nd kinds. |
| $\boldsymbol{q}_r$ | - | Direction vectors $r = (x, y, z)$ |
| $\varrho, \delta$ | - | Permeability anisotropy ratios defined in the text. |
| $\boldsymbol{Q}$ | - | Orthogonal matrix for 3D axes rotation. |
| $\boldsymbol{D}$ | - | Diagonal matrix containing semiaxes of ellipsoids. |
| $\boldsymbol{I}$ | - | Unitary matrix. |
| $\boldsymbol{A}$ | - | coefficient matrix in surface equations of an inertia ellipsoid. |

## Subscripts/Superscripts

| | | |
|---|---|---|
| $i$ | - | inclusion. |
| $s$ | - | skin or ellipsoidal coat in composite inclusion. |
| $m$ | - | matrix. |
| $x, y, z$ | - | component in $x, y, z$ direction. |
| $\infty$ | - | at boundary. |

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